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SYNCHRONIZATION PATTERNS AND RELATED PROBLEMS IN COMBINATORIAL ANALYSIS AND GRAPH THEORY

by

Herbert Taylor

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USCEE TECHNICAL REPORT #509

June 1981

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INTRODUCTION

In the field of Communications and Information distinguishing and identifying are certainly basic operations. Identifying things by their names or distinguishing between different mathematical objects, such as words or messages, by their properties, are operations which need to be restudied when we want to get them done by electronic hardware. It is remarkable how complex such simple considerations can become as, for example, telling whether two small networks are isomorphic or not.

Several conceptual tools which may be of some use to these areas of Electrical Engineering have been gathered together in recent decades under the heading of "Graph Theory". Actually one of the earliest (and one of the best) theorems of what is now called graph theory was discovered and proved by Kirchhoff in the 1847 work [1,2], in which he expounded Kirchhoff's Laws.

Moon [3] gives a detailed history of developments stemming from Kirchhoff's result, typified by the matrix tree theorem, which tells us that the number of spanning trees of a graph can be expressed as the determinant of a matrix whose entries depend on the graph. Apparently Electrical Engineering has already contributed much to the subject of graph theory - perhaps given it its best reason for existing at all.

This paper offers several new theorems about graphs, with the hope that one way or another they will be of some use to Electrical Engineering. Each chapter can be read independently of the others even though the topics are quite similar. An appendix will review the graph theoretic terminology used here.

Chapter 1 contains one new theorem, which can be stated as follows. Let a tree with n nodes be given. If it is possible to label the edges with integers in such a way that the $\binom{n}{2}$ path sums take all the values 1, 2,...,$\binom{n}{2}$, then there must exist an integer m such that $n = m^2$ or $n = m^2+2$.

When such a tree exists it can provide an efficient resistance standard, by making the edge labels be the values of resistors.

Chapter 2 introduces a notation which could be read "A choose B" for graphs A, B. Let $\binom{A}{B}$ equal the number of subsets of nodes of A for which the induced subgraph is isomorphic to B. The following theorem came as a surprise to several competent graph theorists. Let graphs G and H both have girth >k, both have n nodes, and both be regular with the same valence. If F is any forest with total spread $\leq$k, then $\binom{G}{F} = \binom{H}{F}$.

We might find the actual calculation of $\binom{G}{F}$ too difficult in some particular case, but then succeed using H.

Such an application of the theorem is illustrated by the following computational result. Let F be the forest consisting of 4 nodes with no edges. Let G be any regular graph with $n = 2m$ nodes, girth >4, and valence $= 3$. We find that $\binom{G}{F} = \frac{m}{3}(2m^3 - 24m^2 + 100m - 147)$.

Chapter 3 initiates the exploration of a new concept in graph theory, formulated by this writer. When can we be sure of making a choice consisting of one symbol from each node of a graph, with distinct symbols on adjacent nodes? If the same set of symbols is available to choose from at every node, then the question is asking whether the graph is k-colorable. But if we only know that k will be the number of symbols available at every node, and have to face the possibility that different nodes may get different k-sets, then the question is asking whether the graph is k-choosable. If a positive integer f(j) is the number of symbols available at the jth node, then the question is asking whether the graph is f-choosable.

The change of wording, which converts a colorability question into a choosability question, makes a big change in the amount of information needed to answer the question. It is a remarkable combinatorial fact that there is no upper bound to how much the choice number of a graph can differ from the chromatic number. In fact no matter how big k is, there will exist a complete bipartite graph which is 2-colorable but not k-choosable.

The main results of this chapter, presented in nine theorems and seven open questions, are the joint work of this writer with Arthur L. Rubin and Paul Erdös.

Chapter 4 contains a discussion of the m-pire problem on the sphere and on the torus, together with some related problems. Two constructions show that the 3-pire chromatic number of the sphere is 18, and the 4-pire chromatic number of the sphere is 24. Heawood conjectured in 1890 that the m-pire chromatic number of the torus would be 6m+1, for all m. Heawood's conjecture is proved true in this chapter by a construction scheme using an infinite family of graceful triangulations of the sphere which was given by S.W. Golomb in 1972.

An application to VLSI layout problems could be as follows. When a nonplanar graph has to be realized in a flat (planar) design "layout", the usual expedient is to introduce crossings or use several layers. An m-pire scenario would always use two layers, one layer for "connections" and one layer for "nodes", with each node reaching the connection layer in several places.

In Chapter 5 the main object of study is an n×m array of dots and blanks having the property that the absolute vector difference between two dots is never repeated. These are called synchronization patterns because in any position reachable by horizontal and vertical shifting such a pattern will overlap with the original in at most one dot location.

With the number of dots maximized these patterns represent 2-dimensional Golomb rulers, concerning which there are still several open problems. Another type, suited to a sonar application used by Costas, has exactly one dot in each row and column of an $n \times n$ array. Constructions due to L. Welch and A. Lempel provide examples of the Costas type when n is 1 or 2 less than a power of a prime. It is still an open problem to prove that an $n \times n$ pattern with n dots exists for all n (Costas type or Golomb type).

Several applications to radar, sonar, and synchronization problems are discussed with examples of small patterns optimized under different combinations of requirements. An exhaustive enumeration of the Costas type is exhibited up to 6×6.

CHAPTER 1

ODD PATH SUMS IN AN EDGE-LABELED TREE - A PROBLEM DUE TO JOHN LEECH

According to one of many equivalent definitions, a tree is a graph on n nodes in which each of the $\binom{n}{2}$ pairs of nodes is connected by a unique path. Thus if each of the n-1 edges of a tree is labeled with an integer, then each of the $\binom{n}{2}$ pairs of nodes has associated with it a uniquely determined path sum. In the remarkable edge-labeled tree of John Leech (Figure 1-1) the integers have been chosen to make the path sums run consecutively from 1 to $\binom{6}{2}$. It turns out that a key question concerns the number of odd

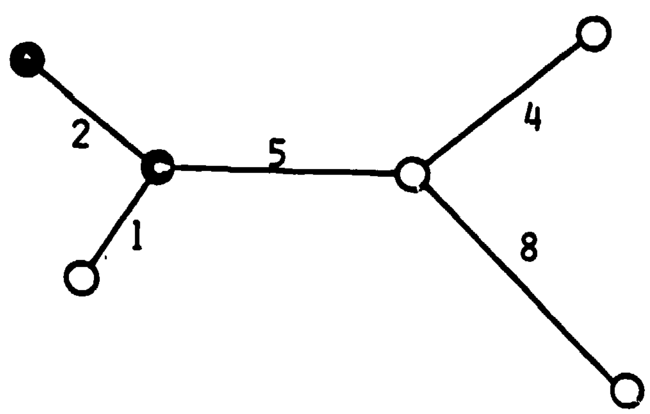


Figure 1-1. The Leech Tree

path sums. A two-coloration trick, introduced below, will show that there are only a few possibilities for that number, over arbitrary labelings of the edges with integers.

Let an edge-labeled tree be given, and choose one node to start with black. Proceed along the edges to all the nodes of the tree, changing black to white or white to black across an odd edge, but keeping the same color across an even edge. When every node has been reached, each odd edge will connect a white node to a black node, and each even edge will connect black to black or white to white. The same property will hold for path sums as well. Indeed, the unique path between two nodes will have an odd sum if and only if it makes an odd number of color changes. Thus any path sum will be odd if and only if the path has a black node at one end and a white node at the other end. That is our two-coloration trick, and it answers our key question as follows: (BW) If the edges of a tree with n nodes are labeled with integers, then the number of odd path sums must be equal to $b \cdot w$, where $b+w = n$.

Evidence that (BW) is non-trivial appears in [4] where John Leech asked for those trees on n nodes which could be edge-labeled to make the path sums take the consecutive values $1, \ldots, \binom{n}{2}$. He gave the complete answer for $n \leq 6$, including the example of Figure 1-1, but said he could offer no information for $n > 6$. By virtue of (BW) we can easily show that it is impossible for certain values of n.

When $\binom{n}{2}$ is even, consecutiveness requires that exactly half of the path sums be odd. In this case we require nonnegative integers b and w such that $b+w = n$ and $2bw = n(n-1)/2$. This reduces quickly to the requirement that $n = (b-w)^2$. When $\binom{n}{2}$ is odd we must have $b+w = n$ and $2bw = n(n-1)/2+1$, which reduces quickly to $n = (b-w)^2+2$. Thus an edge-labeled tree on n nodes cannot have the consecutive path sums $1,\ldots,\binom{n}{2}$ unless, for some integer m, $n = m^2$ or $n = m^2+2$.

Asymptotically speaking, this says that for almost all values of n no tree on n nodes can be labeled as in Figure 1-1. Fan Chung of Bell Labs informed me that Shu Lin did a computer search over all trees with $n = 9$ nodes, and found that none of them could be labeled like Figure 1-1. It is not known whether any exist with $n > 9$ where $n = m^2$ or $n = m^2+2$. I recommend that the reader compare these edge-labeling questions with the node-labeling results and questions to be found in the article [5] by Solomon W. Golomb and I thank him for several conversations and much encouragement.

CHAPTER 2

COUNTING INDUCED SUBGRAPHS

2.1 Introduction

For graphs A,B, let $\binom{A}{B}$ denote the number of subsets of nodes of A for which the induced subgraph is B. If G and H both have girth >k, and if $\binom{G}{T} = \binom{H}{T}$ for every $\leq$ k-node tree T, then for every k-node forest F, $\binom{G}{F} = \binom{H}{F}$. Say the spread of a tree is the number of nodes in a longest path. If G is regular of degree d, on n nodes, with girth >k, and if F is a forest of total spread $\leq$k, then the value of $\binom{G}{F}$ depends only on n and d.

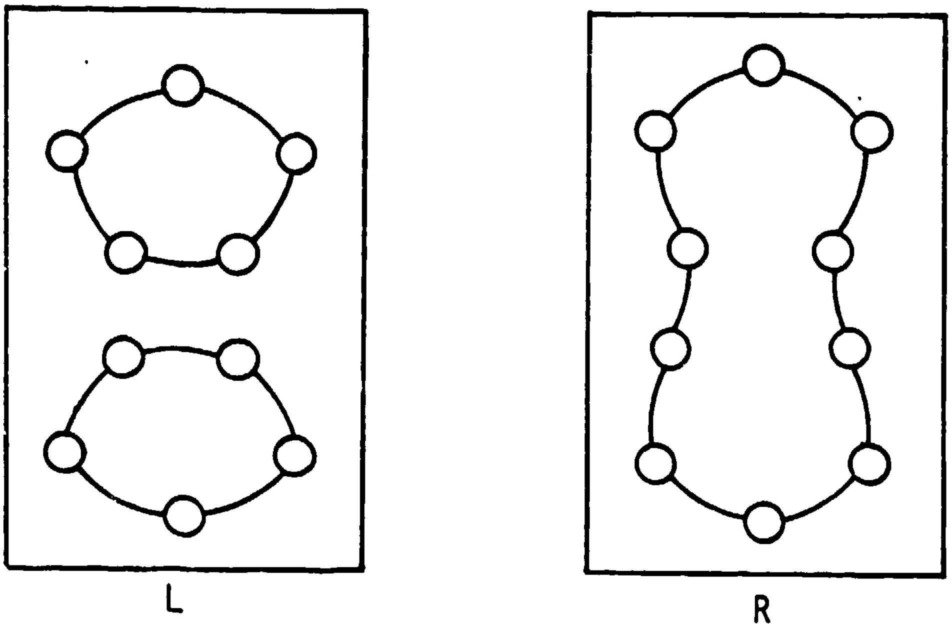


Figure 2-1. The Graph L and the Graph R

In Figure 2-1. L and R are both regular with $d = 2$, that is, 2 edges on each node. Both have girth >4, that is no cycle on ≤ 4 nodes. By direct count $\binom{L}{F} = \binom{R}{F} = 25$, as predicted by our theorem. F is the 4 node forest with no edges.

In Figure 2-2 we have two graphs D and P both regular with $n = 20$, $d = 3$ and girth >4. Again our theorem tells us $\binom{D}{F} = \binom{P}{F}$ for any forest F with 4 nodes, but in this case the direct count to find the number takes more work.

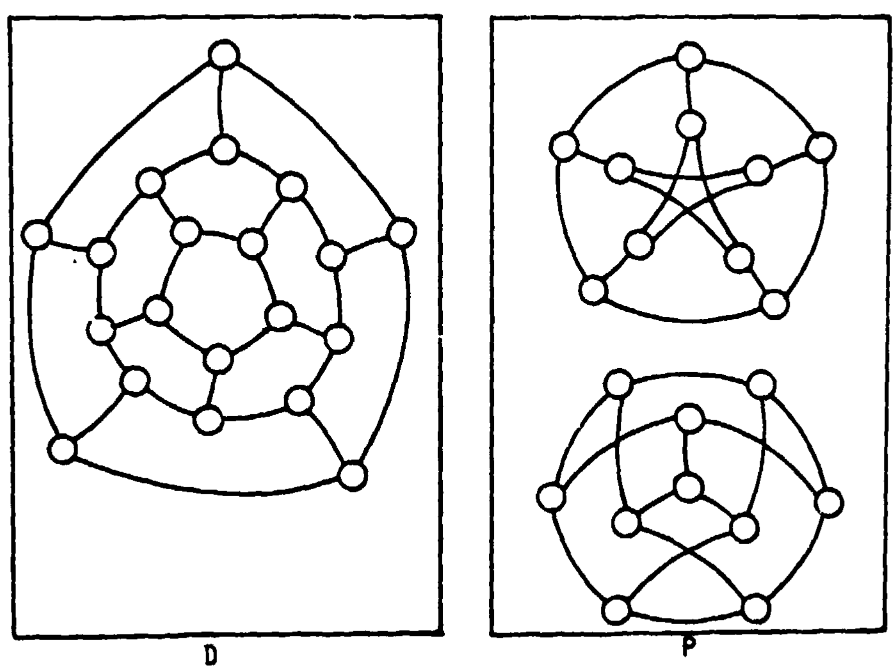


Figure 2-2. The Dodecahedron and Two Copies of the Petersen Graph

In connection with self-checking networks of micro-computers, Simoncini made the following conjecture in 1977. If a graph is regular and has girth >4, then, among all the k-subsets of nodes, the number of k-subsets which induce e edges will depend only on the number n of nodes in the graph, and on the degree of regularity d.

First Simoncini proved it for e = k-1, in which case the induced subgraphs are trees. Working together we verified the conjecture up to k = 5. Later in the fall of 1977 Taylor proved that if any k-node forest F is given, then the number of k-subsets that induce F will depend only on n and d in a graph that is regular and has girth >k. Thus the conjecture is true, and somewhat more.

The lemma was obtained by Taylor early in 1979 -- incorporating the extension, suggested to Schwenk -- to forests with total spread $\leq k$.

2.2 Main Results

Let us say that the spread of a tree is the number of nodes in a longest path. The total spread of a forest should be the sum of the spreads of its connected components (trees). A graph has girth >k iff it has no circuits on $\leq k$ nodes. Two trees in G are adjacent if the union of their nodes induces a connected subgraph of G.

We start with a notation which could be read, "A choose B".

Definition

For graphs A,B let $\binom{A}{B}$ denote the number of subsets of nodes of A for which the induced subgraph is B.

Lemma

Suppose both G and H have girth >k and suppose $\binom{G}{T} = \binom{H}{T}$ for every tree T of spread $\leq k$. Then $\binom{G}{F} = \binom{H}{F}$ for every forest F of total spread $\leq k$.

Proof

We induct on c, the number of components of F. Of course when c = 1, F is a tree, and $\binom{G}{F} = \binom{H}{F}$ by assumption.

For F with c > 1, choose T a component of F, and say $F^* = F-T$. We know inductively that $\binom{G}{F^*} = \binom{H}{F^*}$ also that $\binom{G}{T} = \binom{H}{T}$, and hence that

$$\binom{G}{F^*} \cdot \binom{G}{T} = \binom{H}{F^*} \cdot \binom{H}{T}.$$

Observe that $\binom{G}{F^*}\cdot\binom{G}{T}$ counts the occurrences of a union of one of the $\binom{G}{F^*}$ subsets with one of the $\binom{G}{T}$ subsets. Now we can count those occurrences another way, as follows. Let $W_1,\ldots,W_m$ be the family of distinct forest induced by such unions. We know that the unions induce only forests because of the girth condition. Looking at each forest W_i by itself, let w_i be the number of times it occurs that one of the $\binom{W_i}{F^*}$ in union with one of the $\binom{W_i}{T}$ induces all of W_i. Thereby, summing over i, $\Sigma W_i \binom{G}{W_i}$ counts the occurrences of the unions in G, and similarly $\Sigma W_i \binom{H}{W_i}$ counts them in H.

By two counts we have

$$\Sigma W_i \binom{G}{W_i} = \binom{G}{F*} \cdot \binom{G}{T},$$

and similarly

$$\Sigma W_i \binom{H}{W_i} = \binom{H}{F*} \cdot \binom{H}{T}.$$

We can say $W_1 = F$, and observe that each of $W_2, \ldots, W_m$ has fewer than c components. Inductively for $i \geq 2$ we know that

$$W_i \binom{G}{W_i} = W_i \binom{H}{W_i} .$$

From the above equalities we conclude that

$$W_1 \binom{G}{W_1} = W_1 \binom{G}{W_1}.$$

Of course $W_1 \neq 0$. Thus finally

$$\binom{G}{F} = \binom{H}{F}.$$

<u>Theorem</u>

Suppose G and H both have girth >k, and suppose each has n nodes, and each is regular of degree d.

Then $\binom{G}{F} = \binom{H}{F}$ for every forest F with total spread $\leq$ k.

<u>Proof</u>

In view of the lemma, it will suffice to prove it for every tree T of spread $\leq$k. We induct on the number of nodes of T.

If T consists of a single node, then $\binom{G}{T} = \binom{H}{T} = n$.

For T with more than one node, choose x an end node of T, and say S is the tree T-x. We assume $\binom{G}{S} = \binom{H}{S}$, and must show $\binom{G}{T} = \binom{H}{T}$. Suppose a is the number of nodes of S at which adding an edge to x will reconstitute T. Suppose each such node has b edges of S on it.

Let S_g be one of the $\binom{G}{S}$ subsets. Because G has girth $>k$, and because T has spread $\leq k$, there will be exactly $a\cdot(d-b)$ nodes of G available to increase S_g to a subset counted by $\binom{G}{T}$. We can say that $\binom{G}{S}\cdot a\cdot(d-b)$ is the number of occurrences of S in T in G. The same occurrences are also counted by $\binom{G}{T}\binom{T}{S}$, and so

$$\binom{G}{T}\binom{T}{S} = \binom{G}{S}\cdot a\cdot(d-b).$$

Likewise

$$\binom{H}{T}\binom{T}{S} = \binom{H}{S}\cdot a\cdot(d-b).$$

Of course $\binom{T}{S} \neq 0$.

Thus we conclude $\binom{G}{T} = \binom{H}{T}$, and the proof is complete.

Remarks

Repeating the main step for trees make it easy to prove the following, for G and H as in the theorem.

(i) For trees T and S, with spread T + spread S $\leq$ k, let T_g be one of the $\binom{G}{T}$, and let T_h be one of the $\binom{H}{T}$. Then the number among the $\binom{G}{S}$ adjacent to

T_g is equal to the number among the $\binom{H}{S}$ adjacent to T_h.

Putting (i) together with the lemma makes it easy in turn to prove (ii).

(ii) Let T be one component of a forest F, with total spread $\leq k$, and say $F-T = F^*$ is not null. Let G_t be the graph induced on those nodes of G that are not adjacent to T_g, and let H_t be the graph induced on those nodes of H that are not adjacent to T_h. Then $\binom{G_t}{F^*} = \binom{H_t}{F^*}$.

Discussion

Our results pertain to the braod problem of counting induced subgraphs of various types. For k = 3 the same results could be derived from [6, 8, 9, 10]. Somewhat related problems can be found in [7, 11], listed here just to invite comparison. By way of acknowledgement, we specially wish to thank Allen Schwenk for his very helpful comments.

Lastly, here are the results of two sample calculations.

(1) If F is the k-node forest with no edges, and G is any regular graph with n nodes, girth >k, and d = 2, then

$$\binom{G}{F} = \frac{n}{k} \cdot \binom{n-k-1}{k-1}.$$

(2) If F is the 4 node forest with no edges, and G is any regular graph with $n = 2m$ nodes, girth > 4, and $d = 3$, then

$$\binom{G}{F} = \frac{m}{3}(2m^3 - 24m^2 + 100m - 147).$$

The dodecahedron, D, and one graph, P, consisting of two copies of the Petersen graph, both have 1510 induced copies of F. Thus in Figure 2-2 we have $\binom{D}{F} = \binom{P}{F} = 1510$.

CHAPTER 3

CHOOSABILITY IN GRAPHS

3.1 Introduction

This chapter treats a new concept in graph theory, choosability, and a resulting new parameter, the choice number of a graph.

At the tenth Southeastern Conference on Combinatorics, Graph Theory, and Computing, in April 1979, Jeffrey Dinitz posed the following problem. Given an $m \times m$ array of m-sets, is it always possible to choose one from each set, keeping the chosen elements distinct in every row and distinct in every column? Although simple to state, Dinitz's problem has turned out to be difficult to answer. This writer has proved that the answer is YES for $m \leq 3$, but not heard of any results for $m \geq 4$. It was in the course of working on this problem that the idea arose naturally of letting an adversary put a set of letters on each node of a graph (allowing possibly different sets on different nodes), and then trying to choose a letter from each node, keeping distinct letters chosen from adjacent nodes. What follows represents work done by this writer jointly with Arthur Rubin and Paul Erdös.

Let the nodes be named 1,2,...,n in a graph G. Given a function f on the nodes which assigns a positive integer f(j) to node j, we'll require that the adversary put f(j) distinct letters on node j, for each j from 1 to n. Now we'll say that G is f-choosable if, no matter what letters the adversary puts, we can always make a choice consisting of one letter from each node, with distinct letters from adjacent nodes.

Using the constant function f(j) = k, we'll say that the choice number of G is equal to k if G is k-choosable but not (k-1)-choosable.

For the complete graph with $n \geq 1$, it is always true that the choice number of K_n is equal to n.

Since one of the things the adversary may do is put the same k-set of letters on every node of G, it follows that the choice number = choice #G $\geq \chi G$ = the chromatic number of G. Figure 3-1 shows an example of a graph

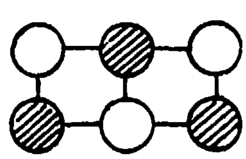


Figure 3-1. A 2-colorable Graph

which is 2-colorable (therefore has chromatic number ≤ 2) but not 2-choosable (therefore has choice number >2). If the adversary uses the pattern pictured in Figure 3-2, then no choice of one letter from each node can have distinct letters from every adjacent pair of nodes.

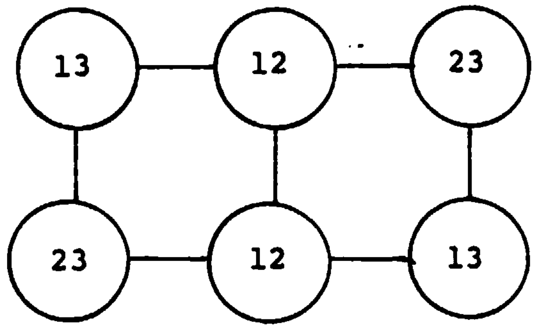


Figure 3-2. The Graph is not 2-choosable

There is no bound on how much choice #G can exceed χG, as n increases. The complete bipartite graph $K_{m,m}$ is 2-colorable. But if $m = \binom{2k-1}{k}$, then $K_{m,m}$ is not k-choosable.

The adversary can construct a pattern to prove it as follows. Recall that $m = \binom{2k-1}{k}$ represents the number of k-subsets of a (2k-1)-set. Picture $K_{m,m}$ with m nodes in the top row, and m nodes in the bottom row, having an edge between two nodes iff one is in the top row, and the other is in the bottom row. Let the letters be the elements of a (2k-1)-set. Put each k-subset of letters on one node of the top row, and on one node of the bottom row.

When we try to make a choice, we find it must include k distinct letters from nodes of the top row - otherwise a k-set consisting of letters not chosen from any node would be the k-subset of letters on some node in the top row. But now the attempted choice must fail in the bottom row because some set of k distinct letters, already chosen from nodes in the top row, will be exactly the k-subset of letters put on some node of the bottom row.

Thus $K_{m,m}$ is not k-choosable, when $m = \binom{2k-1}{k}$.

Figure 3-3 shows the picture when $k = 3$, $m = \binom{5}{3} = 10$, and the set of letters is {1,2,3,4,5}. The dashed line is meant to suggest the 100 edges which connect nodes of the top row with nodes of the bottom row.

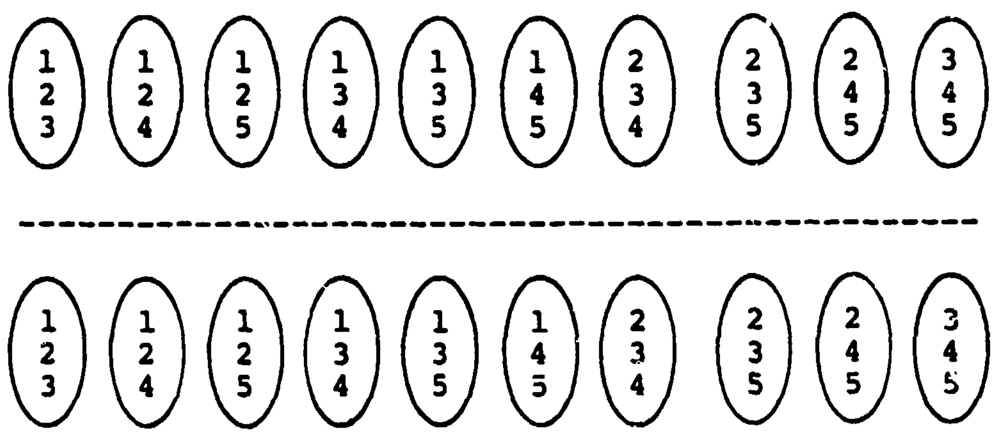


Figure 3-3. $K_{10,10}$ is not 3-choosable

Open Question

What is the minimum number $N(2,k)$ of nodes in a graph G which is 2-colorable but not k-choosable?

Bounds For $N(2,k)$

A family F of sets has property B iff there exists a set B which meets every set in F but contains no set in F. In other words F has property B iff there exists a set B such that

$$\left.\begin{array}{ll} 1. & X \cap B \neq \Phi \\ 2. & X \not\subseteq B \end{array}\right\} \quad \text{for every } X \in F.$$

M_k is defined as the cardinality of a smallest family of k-sets which does not have property B. Although M_k is only known exactly for $k \leq 3$, there are bounds for it. The crude bounds

$$2^{k-1} < M_k < k^2 2^{k+1}$$

will suffice here. Sharper bounds can be found in P. Erdös, "On a Combinatorial Problem III", Canad. Math. Bull., vol. 12, no. 4, 1969.

In what follows we shall prove that

$$M_k \leq N(2,k) \leq 2M_k.$$

To establish the upper bound, we argue that $K_{m,m}$ is not k-choosable when $m \geq M_k$. For the k-sets of letters on nodes of the top row the adversary can use a family F which does not have property B, and use the same F on the bottom row. If C is any set of letters chosen one from each node of the top row, then of course $X \cap C \neq \Phi$ for every $X \in F$, and consequently there must exist $W \in F$ such that $W \subseteq C$. But then in the bottom row no letter can be chosen from the node which has W.

To establish the lower bound, we argue that $K_{b,t}$ is k-choosable when $b+t < M_k$. With t nodes in the top row, and b nodes in the bottom row, let F be the family of k-sets of letters assigned to the nodes. F will have property B because $|F| < M_k$, and we can use B to make our choices. First choose a letter of B from each node in the top row - the choice exists because B meets each of them. Then choose a letter not in B from each node of the bottom row - that choice exists because B does not contain any of them. Two nodes are adjacent only when one is in the top row, and the other in the bottom row, and their chosen letters are distinct because one is in B, and the other is not in B.

That completes the proof. The following theorem summarizes the above discussion.

<u>Theorem</u>

$$2^{k-1} < M_k \leq N(2,k) \leq 2M_k < k^2 2^{k+2}.$$

Here is all we know regarding exact evaluation of $N(2,k)$.

$$M_1 = 1 \qquad N(2,1) = 2$$

$$M_2 = 3 \qquad N(2,2) = 6$$

$$M_3 = 7 \qquad 12 \le N(2,3) \le 14$$

Although it is most likely that $N(2,3) = 14$, it would be quite a surprise if $N(2,k) = 2M_k$ were to persist for large k. We know that $M_k+1 < N(2,k)$, for $k > 1$.

$K_{7,7}$ is pictured below with the adversary's assignment which shows it is not 3-choosable. Again the dashed line indicates the 49 edges.

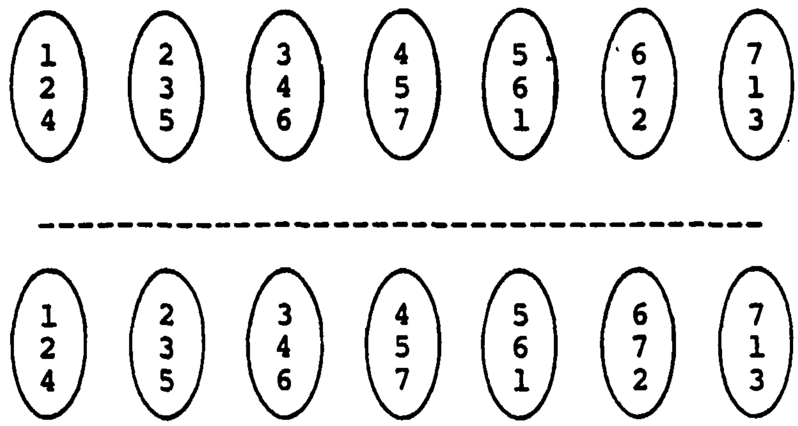


Figure 3-4. $K_{7,7}$ is not 3-choosable

3.2 Characterization of 2-choosable Graphs

A graph is 2-choosable iff each connected component is 2-choosable, so we restrict our attention to connected graphs. To start the investigation of which graphs are 2-choosable, consider a node of valence 1. We can always choose one of its two letters after deciding which letter to choose from the one node adjacent. The obvious thing to do is prune away nodes of valence 1, successively until we reach the core, which has no nodes of valence 1. A graph is 2-choosable iff its core is 2-choosable.

By definition let's say a Θ graph consists of two distinguished nodes i and j together with three paths which are node disjoint except that each path has i at one end, and j at the other end. Thus a Θ graph can be specified by giving the three paths' lengths. Figure 3-5 shows some examples.

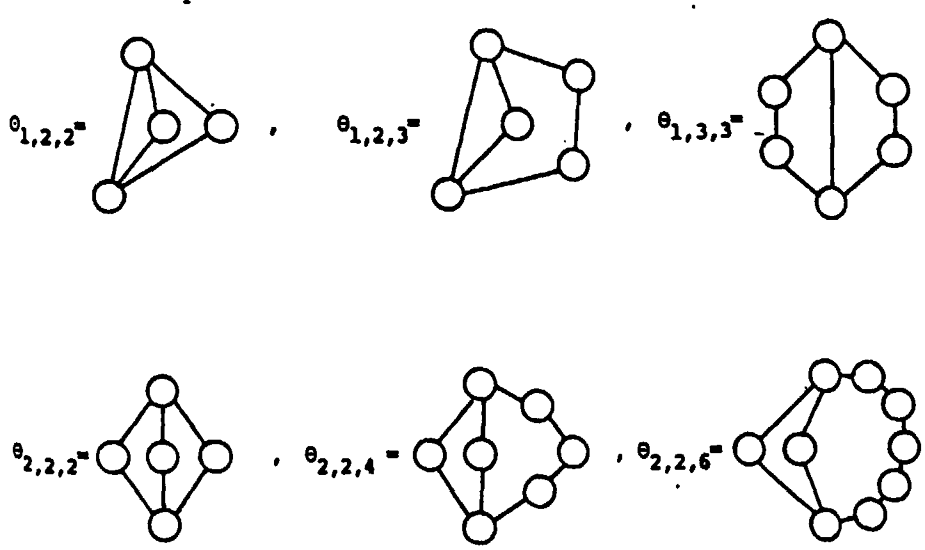


Figure 3-5. Examples of Θ Graphs

Here is a proof that $\Theta_{2,2,2m}$ is 2-choosable, for $m \geq 1$. Let the assigned 2-sets be named as in the picture.

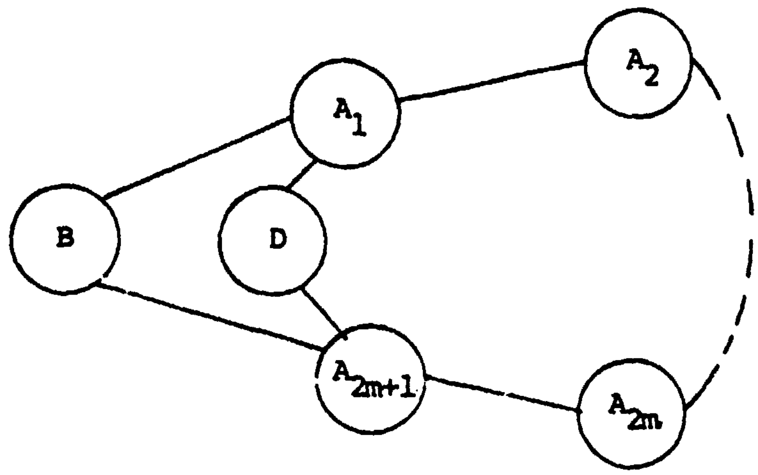


Figure 3-6. Naming 2-sets

CASE I: Suppose $A_1 = A_2 = \ldots = A_{2m+1} = \{x,y\}$. From A_i choose x when i is odd, y when i is even, so that x is chosen from both A_1 and A_{2m+1}. Complete the choice with a letter from $B-\{x\}$, and a letter from $D-\{x\}$.

CASE II: Suppose the A_j's are not all equal. Find one particular adjacent pair $A_i \neq A_{i+1}$. <u>Tentatively</u> choose $x_i \in A_i - A_{i+1}$, and go in sequence choosing $x_{i-1} \in A_{i-1} - \{x_i\}$, $x_{i-2} \in A_{i-2} - \{x_{i-1}\}$, ... until $x_1 \in A_1 - \{x_2\}$. At this point we look ahead to $A_{2m+1} = \{b,d\}$, and look at B and D. If $\{B,D\} \neq \{\{x_1,b\},\{x_1,d\}\}$, then there will exist a choice of $x \in A_{2m+1}$ such that $B-\{x_1,x\} \neq \Phi$ and $D-\{x_1,x\}\neq\Phi$, and so we can continue choosing $x_{2m} \varepsilon A_{2m} - \{x\}$, $x_{2m-1} \varepsilon A_{2m-1} - \{x_{2m}\}$, ... until $x_{i+1} \varepsilon A_{i+1} - \{x_{i+2}\}$, thereby completing the choice. But if $\{B,D\} = \{\{x_1,b\}, \{x_1,d\}\}$, then we go back to $A_i \neq A_{i+1}$ and start the other way. Start by choosing $y_{i+1} \varepsilon A_{i+1} - A_i$, and go in sequence choosing $y_{i+2} \varepsilon A_{i+2} - \{y_{i+1}\}, \ldots$ until $y \varepsilon A_{2m+1} - \{y_{2m}\}$. Here $y \neq x_1$ so we can choose $x_1 \varepsilon B$ and $x_1 \varepsilon D$, and continue with $y_1 \varepsilon A_1 - \{x_1\}$, $y_2 \varepsilon A_2 - \{y_{i-1}\}$ ... until we complete the whole choice at $y_i \varepsilon A_i - \{y_{i-1}\}$.

That completes the proof that $\Theta_{2,2,2m}$ is 2-choosable.

Since an even cycle C_{2m+2} is a subgraph of $\Theta_{2,2,2m}$, we also know that all even cycles are 2-choosable.

At this point in the investigation, every 2-choosable graph we know about has as its core a subgraph of some $\Theta_{2,2,2m}$. The remarkable fact that no others exist will be told as follows.

Let $\{K_1, C_{2m+2}, \Theta_{2,2,2m} : m \geq 1\} = T$.

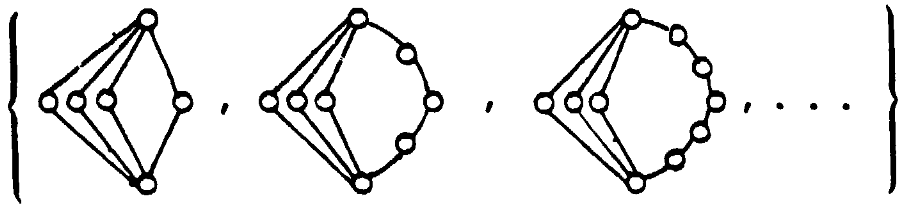
Theorem (A.L. Rubin)

A graph G is 2-choosable if, and only if, the core of G belongs to T.

Proof

Let G be the core of a connected graph.

The idea of the proof is to show, by exhausting the possibilities, that either G is in T, or else G contains a subgraph belonging to one of the following five types.

1. An odd cycle.
2. Two node disjoint even cycles connected by a path.
3. Two even cycles having exactly one node in common.
4. $\Theta_{a,b,c}$ where $a \neq 2$ and $b \neq 2$.
5. 

We start by assuming that G is not in T.

If G contains an odd cycle we are done. Thus we proceed on the assumption that G is bipartite.

Let C_1 be a shortest cycle. Note that there must exist an edge of G not in C_1, because otherwise G would be an even cycle.

If there is a cycle C_2 having at most one node in common with C_1, then we will be in case (2.) or (3.), and be done.

Let P_1 be a shortest path, edge disjoint from C_1, and connecting two distinct nodes of C_1. (This is now known to exist.)

If $C_1 \cup P_1$ is not in T, then it must be in case (4.), in which case we are done.

Now suppose $C_1 \cup P_1$ is in T, so it must be a $\Theta_{2,2,2m}$, and C_1 must be a 4-cycle. Observing that there must be more to G, we can say the following.

Let P_2 be a shortest path, edge disjoint from $C_1 \cup P_1$, connecting two distinct nodes of $C_1 \cup P_1$.

Next we examine six cases to see what the end nodes of P_2 might be. It will help to name the nodes of C_1 as shown in this picture of $C_1 \cup P_1$.

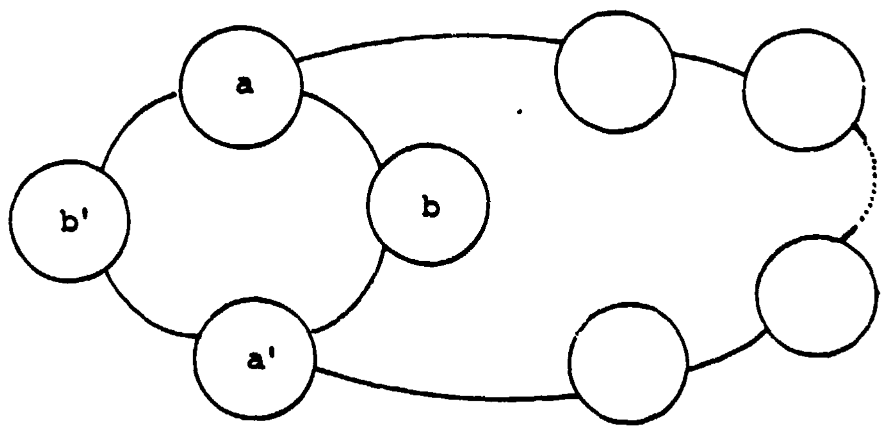


Figure 3-7. Picture of $C_1 \cup P_1$

Case (i). If the ends of P_2 are two interior nodes of P_1, then we have a cycle disjoint from C_1, and are in case (2.) again.

Case (ii). If the ends of P_2 are a and an interior node of P_1, then we have a cycle with exactly one node in common with C_1, and are in case (3.).

Case (iii). If the ends of P_2 are b and an interior node of P_1, then we have a path from a to b edge disjoint from C_1, which puts us in case (4.).

Case (iv). If the ends of P_2 are a and b, we are put in case (4.) again, as we were in case (iii).

Case (v). If the ends of P_2 are a and a', and P_1 is of length 2, then we are in case (5.). If P_1 is of length >2, then we are in case (4.).

Case (vi). If the ends of P_2 are b and b', then by removing any edge of C_1 we find a Θ graph which puts us in case (4.).

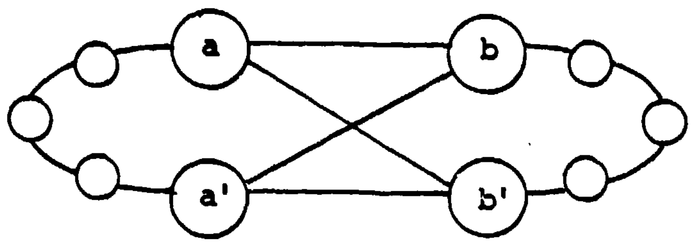


Figure 3-8. Illustrating Case (vi)

We now know that if G is not in T, then G contains one of the five types. Thus it only remains to show that any graph of type 1., 2., 3., 4., or 5. is not 2-choosable.

Type (1.) is not even 2-colorable.

To deal with 2., 3., 4., 5. we can use the following reduction. Remove a node b, and merge the nodes that were adjacent to b. Any multiple edges that result can be made single, and no "loops" will appear, because the graph remains bipartite. If the reduced graph G' is not 2-choosable, then G is not 2-choosable.

To prove it, suppose G' is not 2-choosable. Unmerge, and assign the same {x,y} to b as to all the nodes adjacent to b. If, say, x is the letter chosen from b, then y will have to be chosen from all the nodes adjacent to b, and therefore a choice for G would have worked just as well for G'. It is worth special notice that this proof would not have worked for 3-choosability.

After repeated application of this reduction process, we will only need to verify that each of the four particular graphs shown in Figure 3-9 is not 2-choosable.

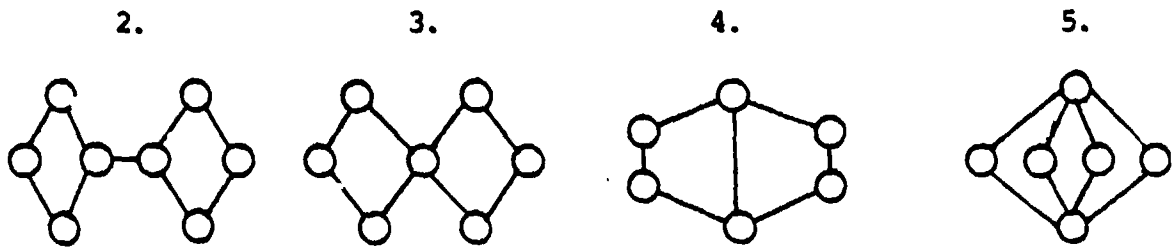


Figure 3-9. Four Particular Graphs

No choice exists for the assignments shown in Figure 3-10.

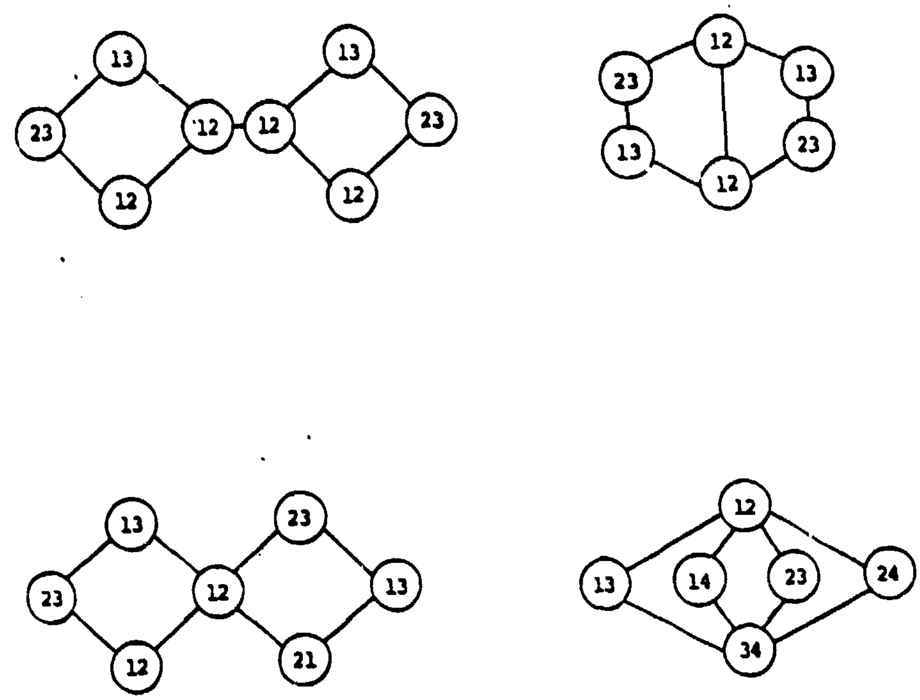


Figure 3-10. None of the Four Particular Graphs is 2-choosable

That completes the proof of the theorem characterizing 2-choosable graphs.

3.3 A Theorem on Graph Structure

The following theorem is due to Arthur Rubin. It will lead to a characterization of D-choosability, and consequently to a generalization of Brooks' theorem. But, apart from choosability considerations, here is a remarkable theorem.

Theorem R

If there is no node which disconnects G, then G is an odd cycle, or $G = K_n$, or G contains, as a node induced subgraph, an even cycle without chord or with only one chord.

Proof (By exhaustion, and induction on n)

Assume no node disconnects G, G is not an odd cycle, and $G \neq K_n$. Observe that a Θ graph either contains an even cycle as a (node) induced subgraph, or consists of an even cycle with only one chord. Thus each subcase will be settled when we find an induced even cycle in G, or find an induced Θ graph in G.

CASE I. There is a node of valence 2. Call it N. Remove N, and prune nodes of valence 1 successively. Now look at what is left.

I.1 One node. G must have been a cycle (not odd).

I.2 An odd cycle. G must have been a Θ graph.

I.3 K_m, where $m \geq 4$. We find an induced $\Theta_{1,2,p}$, where p is the length of the pruned off path.

I.4 If I.1, I.2, I.3 do not hold, and still the graph that remains after pruning has no node which disconnects it, then we're done by the inductive hypothesis.

I.5 What remains has a node X which disconnects it. Name the end nodes of the pruned off path A and B. First we argue that A could not disconnect what remains, because contrariwise it would have to have done so before pruning as well. Thus we know $A \neq X \neq B$.

What if A and B were connected by some path not through X? If this were so, then X would have disconnected G before pruning. Thus all paths from A to B go through X.

Let α be a shortest path from A to B. The picture should look something like the one in Figure 3-11. Naturally a shortest path cannot have any chords.

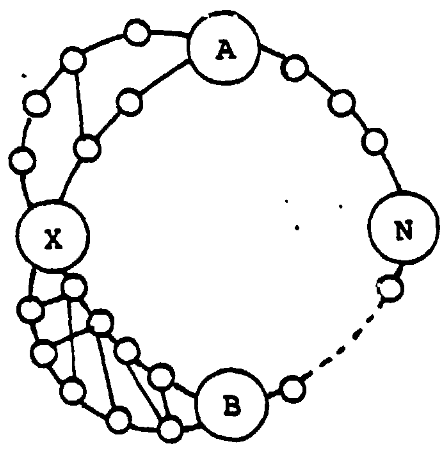


Figure 3-11. Typical for Case I.5

Let β be a shortest path from a node U adjacent to A (U not on α, A not on β), to a node Z adjacent to α-A.

If Z is adjacent to more than one node of α-A, let Y_1 and Y_2 be the two such closest to A along α. Then the nodes on the arc of α from A to Y_2, and on β, induce a cycle with only one chord, that is, a Θ graph.

If Z is adjacent to only one node of α-A, then the nodes of α,β, and the path through N induce a Θ graph.

CASE II. There is no node of valence 2. Delete one node N, and look at what is left.

II.1 It cannot be just one node.

II.2 An odd cycle γ. Note first that N must have been adjacent to every node of γ. If γ were a 3-cycle, then G would have been K_4. Thus γ is a larger cycle, and we find the "diamond", $\theta_{2,1,2}$, induced in G. It will look like Figure 3-12.

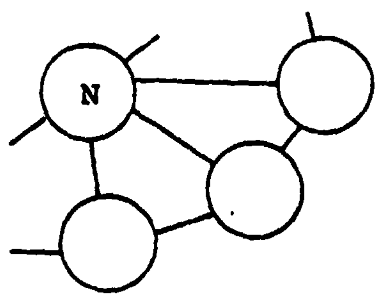


Figure 3-12. Sample for Case II.2

II.3 If G-N is a complete graph, then since G is not K_n, there must be some node Y of G-N which is not adjacent to N. In this case we find a diamond induced in G.

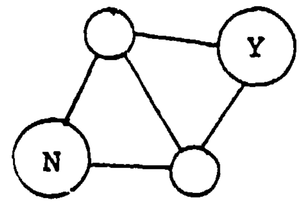


Figure 3-13. Diamond in Case II.3

II.4 If not II.2, not II.3, and G-N has no node which which disconnects it, then we're done by the inductive hypothesis.

II.5 Otherwise the graph G-N has a node X which disconnects it.

First we observe that the subgraph induced on nodes adjacent to N cannot be a complete graph. If it were, then the node X which disconnects G-N would also disconnect G.

Let α be a shortest path in G-N between two nodes A and B which are adjacent to N, but not themselves adjacent.

If the number of edges of α is equal to 2, then we have C_4 or $\Theta_{2,1,2}$, in the form of Figure 3-14.

Otherwise α has more than two edges, and we construct as follows.

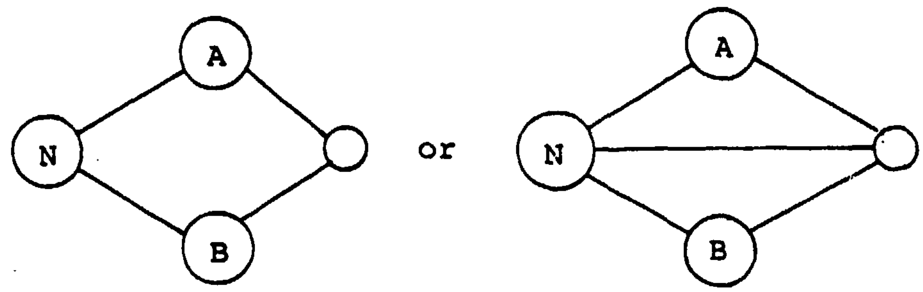


Figure 3-14. Case II.5

Let β be a shortest path in G-N, from a node C which is different from A and B but adjacent to N, to a node Z which is adjacent to α. (C = Z is possible).

In case Z is adjacent to two or more nodes of α, we can identify two more nodes, as follows.

Let Y_A be adjacent to Z, along α, closest to A.

Let Y_B be adjacent to Z, along α, closest to B.

The picture of Figure 3-15 may help remember the above adjacencies.

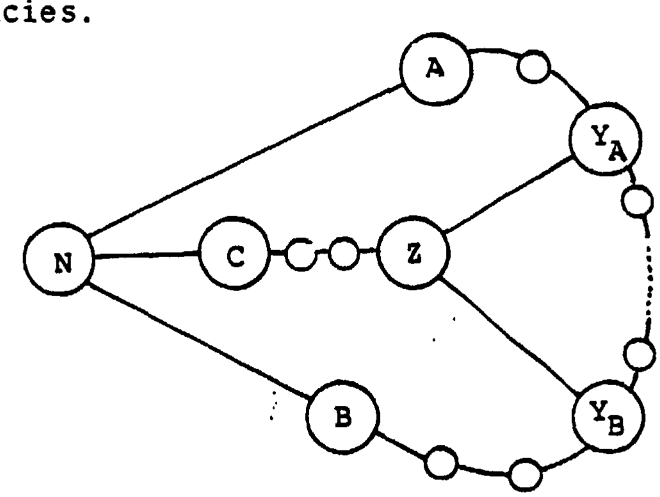


Figure 3-15. Case II.5

If Y_A is not adjacent to Y_B, then the Θ graph we find is the induced subgraph on N, β, the arc of α from A to Y_A, and the arc of α from B to Y_B.

If Y_A is adjacent to Y_B, and $Y_B \neq B$, then our Θ graph is induced on N, β, and the arc of α from A to Y_B.

If Y_A is adjacent to Y_B, then $Y_B = B$, then $Y_A \neq A$, so it is symmetric with the previous case.

Finally, if Z is adjacent to only one node of α, then our Θ graph is induced by N, α, and β.

The proof of theorem R is complete.

3.4 Characterization of D-choosability

To define a function D on the nodes of G, let D(j) = the valence of node j.

Thus the question of whether G is D-choosable, or not, is posed by specifying that the number of letters assigned to a node shall be equal to the number of edges on that node. We start by exploring graphs which are not D-choosable.

Supposing G and H are two separate graphs, take any node i of G, and any node j of H, and merge them into a single node (ij) to produce a new grah G (ij) H. It goes understood that the node (ij) disconnects G (ij) H.

Generate a family non D as follows. For every integer $n \geq 1$, put K_n into non D. Put all odd cycles into non D.

Whenever G ε non D and H ε non D, put G (ij) H into non D. A typical member of the family non D will look like Figure 3-16.

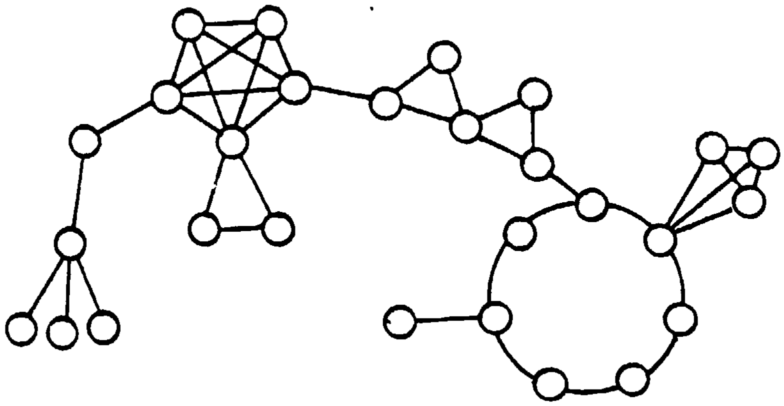


Figure 3-16. Typical Non D

Since all complete graphs and odd cycles are not D-choosable, it will become apparent that every graph in non D is not D-choosable, after we prove a quick lemma.

Lemma

If G and H are both not D-choosable, then G (ij) H is not D-choosable.

Proof

Presume the adversary's assignments used different letters on G . H. Let A be the set of letters put on node i of G, and let B be the set of letters put on node j of H. Since D(ij) = D(i) + D(j), the adversary can assign $A \cup B$ to the node ij of G ij H, and keep the other assignments as before the merger. When we try to choose a letter from ij , our choice will fail in G if we take a letter from A, and fail in H if we take a letter from B.

Next we explore graphs which are D-choosable, starting with Θ graphs.

Consider an arbitrary $\Theta_{a,b,c}$ with say, $c \geq 2$. Let the nodes be named 1,2,...,n as shown in the picture Figure 3-17.

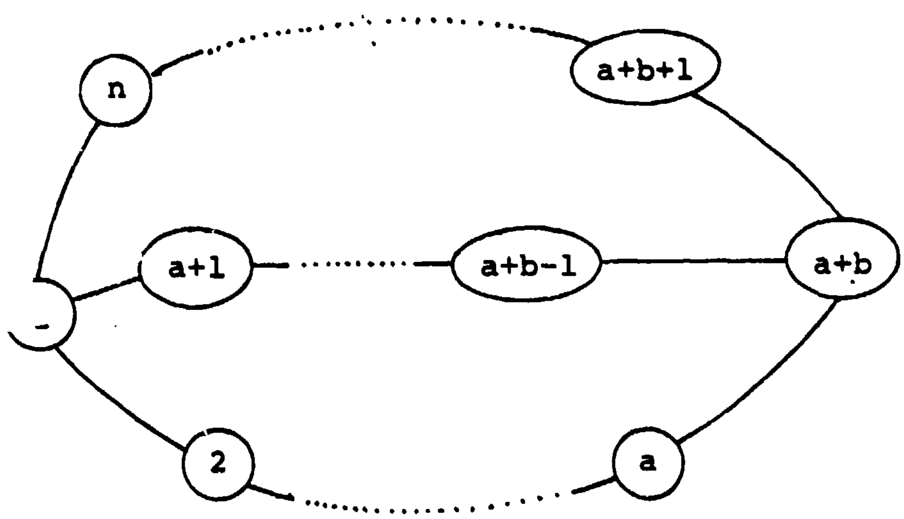


Figure 3-17. Arbitrary Θ Graph

Make the choices in sequence, starting at node 1. Node 1 has three letters, so we can choose a letter not in node n. At each node in sequence there will be more letters than adjacent earlier nodes, until we reach node n. Node n is adjacent to two earlier nodes, but neither of its two letters is excluded by the choice we made at node 1. Thus every Θ graph is D-choosable. Also, recalling our discussion of 2-choosability, we know that every even cycle is D-choosable.

Lemma

If G is connected, and G has an induced subgraph H which is D-choosable, then G is D-choosable.

Proof

Assuming G-H is not empty, find a node x, of G-H, which is at maximal distance from H. This guarantees that G-x will be connected. Start the choice with any letter from x, and then erase that letter from all nodes adjacent to x. The choice can be completed because G-x is an earlier case.

Theorem

Assume G is connected. G is not D-choosable iff G ε non D.

Proof

Take G and look at parts not disconnected by a node. If every such part is an odd cycle or a complete graph, then G ε non D, and therefore G is not D-choosable.

If some such part is neither an odd cycle nor a complete graph, then Theorem R tells us that G must contain, as a node induced subgraph, an even cycle or a particular kind of Θ graph. By the preceding lemma this means that if $G \notin$ non D, then G is D-choosable.

<u>Same Theorem</u>

Assume G is connected. G is D-choosable iff G contains an induced even cycle or an induced Θ graph.

<u>Comment</u>

As a consequence of this characterization, we can prove that, for large n, almost all graphs are D-choosable.

3.5 <u>Digression → Infinite Graphs</u>

Consider the infinite asterisk in Figure 3-18.

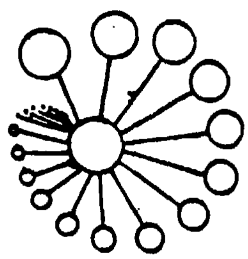


Figure 3-18. The Infinite Asterisk

It is not D-choosable, because the adversay can use Z^+, the set of positive integers, thus

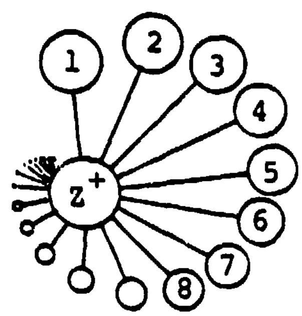


Figure 3-19. Not D-choosable

On the other hand, if we disallow infinitely many edges on any one node, we get the following.

Theorem

Let G be a countably infinite connected graph with finite valence. Then G is D-choosable.

Proof

Let the nodes of G be named with the positive integers. At each node the number of letters put there by the adversary will be no less than the number of edges on that node. Choose letters by the following rules - with i the smallest named node from which a letter has not yet been chosen.

1. If erasing i does not leave a finite component disconnected from the rest of the graph, choose a letter x from node i. Erase x from the nodes adjacent to i, and remove i from further consideration.

2. If erasing i would disconnect a finite component, deal with each such finite component before dealing with i. In each finite component start with a node at maximal distance from i, to be sure it will not disconnect the component.

By following rules 1. and 2. we can choose a letter from node j for every $j \in Z^+$, and never take the same letter from two adjacent nodes.

3.6 Corollary: Brooks' Theorem

The infinite case of Brooks' Theorem is an immediate consequence of the theorem just proved. The finite case is a consequence of Rubin's characterization of D-choosability. Refer to R.L. Brooks, "On Colouring the Nodes of a Network", *Proc. Cambridge Philosophical Soc.*, vol. 37 (1941).

Here is the statement of his original theorem, verbatim:

> Let N be a network (or linear graph) such that at each node not more than n lines meet (where n > 2), and no line has both ends at the same node. Suppose

also that no connected component of N is an n-simplex. Then it is possible to colour the nodes of N with n colours so that no two nodes of the same colour are joined.

An n-simplex is a network with n+1 nodes, every pair of which are joined by one line.

N may be infinite, and need not line in a plane.

Of course for D-choosable graphs, Brooks' theorem holds a fortiori.

Now consider $G \varepsilon$ non D. Pick one node j of G, and define a new function ${}^{j}D$ thus: Let ${}^{j}D(j) = 1+D(j)$, and let ${}^{j}D(i) = D(i)$ if $i \neq j$. We can see that G is D-choosable by attaching an infinite tail at j, as in Figure 3-20.

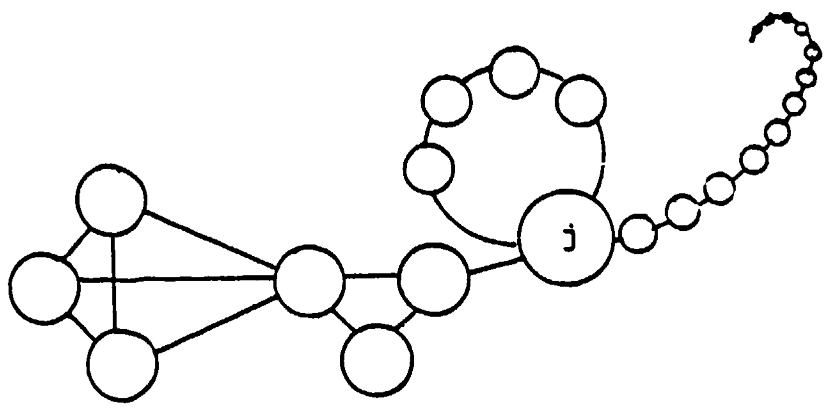


Figure 3-20. Infinite Tail Attached

Lastly, with the observation that the only regular (D(i) = constant) graphs in non D are complete graphs or odd cycles, we have a choice version which covers the finite case.

Theorem

If a connected graph G is not K_n, and not an odd cycle, then choice #G $\leq$ max D(j).

END DIGRESSION

According to a well known result of Nordhaus and Gaddum, $XG + XG \leq n+1$. Before proving the choice version, we state a lemma which may prove useful elsewhere.

3.7 A Choosing Function Lemma

Let the nodes of G be labeled 1,2,...,n, as usual. In that order define a choosing function g, as follows.

$$g(j) = 1 + |\{i : 1 \leq i < j \leq n, \text{ and } \{i,j\} \text{ is an edge of } G\}|.$$

A choosing function has four immediate properties.

1. G is g-choosable.
2. choice #G $\leq$ max g(j) $\qquad 1 \leq j \leq n$
3. $g(j) \leq j$
4. $g(j) \leq 1 +$ G valence j

Theorem

Choice #G + choice #$\overline{G}$ $\leq$ n+1

Proof

Label the nodes 1,2,...,n in such a way that G valence i $\geq$ G valence j if i $\leq$ j. Let g be the choosing function which results from that labeling. Let $\bar{g}$ be the reverse in $\bar{G}$ defined by $\bar{g}(i) = 1+|\{j:1 \leq i < j \leq n$, and $\{i,j\}$ is an edge of $\bar{G}\}|$.

Properties 1., 2., and 4. still hold for $\bar{g}$ and $\bar{G}$, while

$\bar{3}$. $\bar{g}(i) \leq n+1-i$.

Observe that, because of the special labeling, G valence j + $\bar{G}$ valence i $\leq$ n-1, whenever j $\geq$ i. When j $\leq$ i, we have $g(j) + \bar{g}(i) \leq j+n+1-i \leq n+1$. When j $\leq$ i, we have $g(j)+\bar{g}(i) \leq 1+$G valence j + 1 + $\bar{G}$ valence i $\leq$ n+1.

Hence max $g(j)$ + max $\bar{g}(i) \leq n+1$.

The proof is finished by property 2.

3.8 The Random Bipartite Choice Number

Now we present a theorem which tells that there exist constants C_1 and C_2 such that an m×m random bipartite graph will have choice number between $C_1 \log m$ and $C_2 \log m$. The proof will be self contained, with the aid of a lemma.

Having fixed m top nodes, and m bottom nodes, let $R_{m,m}$ denote any one of the bipartite graphs whose edges constitute a subset of the m^2 possible top-to-bottom edges. We think of $R_{m,m}$ as having been chosen at random. Also we think of a "txt" as any pair consisting of a t-subset of top nodes and a t-subset of bottom nodes.

Lemma

Suppose $t \geq \frac{2 \log m}{\log 2}$, and let $\overline{E}$ be the event that an $R_{m,m}$ has an empty induced subgraph on at least one txt.

Then $\overline{E}$ has probability $< \frac{1}{(t!)^2}$.

Proof of Lemma

The number of possible $R_{m,m}$'s is 2^{m^2}. The number of txt's is $\binom{m}{t}^2$. Each possible edge empty txt is contained in $2^{m^2-t^2}$ of the $R_{m,m}$'s. Thus, the number of $R_{m,m}$'s which contain at least one empty txt is $< 2^{m^2-t^2}\binom{m}{t}^2$.

Therefore $\overline{E}$ has probability $< \frac{2^{m^2-t^2}\binom{m}{t}^2}{2^{m^2}} = \frac{\binom{m}{t}^2}{2^{t^2}}$.

With $m \leq 2^{\frac{t}{2}}$, we calculate as follows.

$$\frac{\binom{m}{t}^2}{2^{t^2}} \leq \frac{\binom{2^{t/2}}{t}^2}{2^{t^2}} \leq \frac{\left(\frac{(2^{t/2})^t}{t!}\right)^2}{2^{t^2}} = \frac{1}{(t!)^2} .$$

Theorem

Suppose $\frac{\log m}{\log 6} > 121$, and $t = \left\lceil \frac{2 \log m}{\log 2} \right\rceil$. Then with probability $> 1 - \frac{1}{(t!)^2}$, we have

$$\frac{\log m}{\log 6} < \text{choice \# } R_{m,m} < \frac{3\log m}{\log 6}$$

Proof

For the upper bound, we know from the discussion of $N(2,k)$ that if $2^{k-3} < m \leq 2^{k-2}$, then choice $\#K_{m,m} \leq k$. This tells us that choice $\#R_{m,m} \leq$ choice $\#K_{m,m} < \frac{\log m}{\log 2} + 3 < \frac{3\log m}{\log 6}$.

To derive the lower bound, let $k = \left\lfloor \frac{\log m}{\log 6} \right\rfloor > 120$. Using the fact that $e^k > k^k/k!$, and a calculator if necessary, we obtain:

$$m \geq 6^k > 7k^2 2^k e^k > 7k^2 \left(\frac{2^k k^k}{k!}\right) > 7k^2 \binom{2k-1}{k} > t \cdot k \cdot \binom{2k-1}{k}.$$

Harmlessly supposing $m = t \cdot k \cdot \binom{2k-1}{k}$ we next describe an assignment of letters the adversary can use to show that almost all $R_{m,m}$ have choice number $>k$. $\binom{2k-1}{k}$ is the number of k-subsets of letters from $\{1,2,\ldots,2k-1\}$. Each k-subset is put on k t of the top nodes, and likewise on the bottom nodes. Now consider what must happen when a choice is attempted.

First we argue that on top there must be $\geq k$ letters, each chosen from $\geq t$ nodes. Because otherwise, if $\leq k-1$ letters were chosen $\geq t$ times each, we could look at a k-subset of remaining letters. That k-subset was put on k·t top nodes, therefore one of the letters in it must have been chosen $\geq$ k·t/k = t times.

Similarly there must be $\geq$k letters, each chosen from $\geq$t bottom nodes. But then, since only 2k-1 letters were used, there must be one letter simultaneously chosen from t top nodes and t bottom nodes. The attempted choice fails if this txt has an edge. Now, according to the lemma, in almost all possible $R_{m,m}$'s every txt does have an edge. Thus the lemma tells us that choice $\#R_{m,m} > k$, with probability $> 1 - \frac{1}{(t!)^2}$. In other words, we have proved the lower bound:

with probability $>1 - \frac{1}{(t!)^2}$, we have choice $\#R_{m,m} > \frac{\log m}{\log 6}$.

3.9 The Random Complete Graph - Open Questions

We do not know good bounds for the choice number of the random complete graph. Having fixed n nodes, let R_n denote any one of the graphs whose edges constitute a subset of the $\binom{n}{2}$ possible edges. We think of R_n as having been chosen at random, and look for bounds L(n) and U(n) for which we can prove that

$$L(n) < \text{choice } \#R_n < U(n),$$

with probability $\rightarrow$ 1 as n gets large.

From the known bounds for χR_n, we know that there exists a constant c such that $\frac{cn}{\log n} \leq L(n)$. On the upper side we merely know that $U(n) \leq \frac{n}{2}$.

Thus a specific open problem is to prove that with probability $\to 1$, $\frac{\text{choice } \#R_n}{n} \to 0$ as $n \to \infty$.

It would be even better to find good bounds for $K_{m*r} = K_{m,m,\ldots,m}$ where the number of nodes is $n = rm$, and m is about the size of log n.

We do know that choice $\#K_{3*r} > \frac{4}{3} r+c$.

The only one of this kind for which we know the exact value is K_{2*r}, which may be of interest because it is the only example we have whose proof uses the P. Hall theorem.

<u>Theorem</u>

Choice $\#K_{2*r} = r$.

<u>Proof</u>

Starting at $r = 2$, we already know that choice $\#K_{2*2} = 2$. (it is the 4-cycle).

To induct, suppose $r > 2$, and suppose we know the theorem for all cases $<r$. Let the adversary put r letters on every node. If some letter is on both nodes of a nonadjacent pair, we can choose that letter from both nodes of that pair, and delete it from all other nodes. We can complete the choice by induction in this case.

Otherwise every nonadjacent pair has a disjoint pair of sets of letters. Any union of $\leq r$ of the sets of letters on nodes will have $\geq r$ letters. Any union of $>r$ of the sets will have $\geq 2r$ letters, because it will include a disjoint pair. The conditions for the P. Hall theorem are satisfied, so there exists a system of distinct representatives. That is, the choice exists.

Here are some more specific numbers which are easily proved.

$K_{k-1,m}$ is k-choosable for all m, all $k \geq 2$.

$$K_{k,m} \quad \text{is} \begin{cases} \text{k-choosable for } m < k^k \\ \text{not k-choosable for } m \geq k^k. \end{cases}$$

3.10 Planar Graphs

Since every planar graph has a node of valence ≤ 5, it follows easily that every planar graph is 6-choosable. Perhaps some mathematicians, who are dissatisfied with the recent computer proof of the 4-color theorem, still sense that there are some things we ought to know, but do not yet know, about the structure of planar graphs. Here we offer two conjectures which may incidentally add interest to that exploration.

Conjecture

Every planar graph is 5-choosable.

Conjecture

There exists a planar graph which is not 4-choosable.

Question

Does there exist a planar bipartite graph which is not 3-choosable?

Figure 3-21 shows a graph which is planar, and 3-colorable, but not 3-choosable.

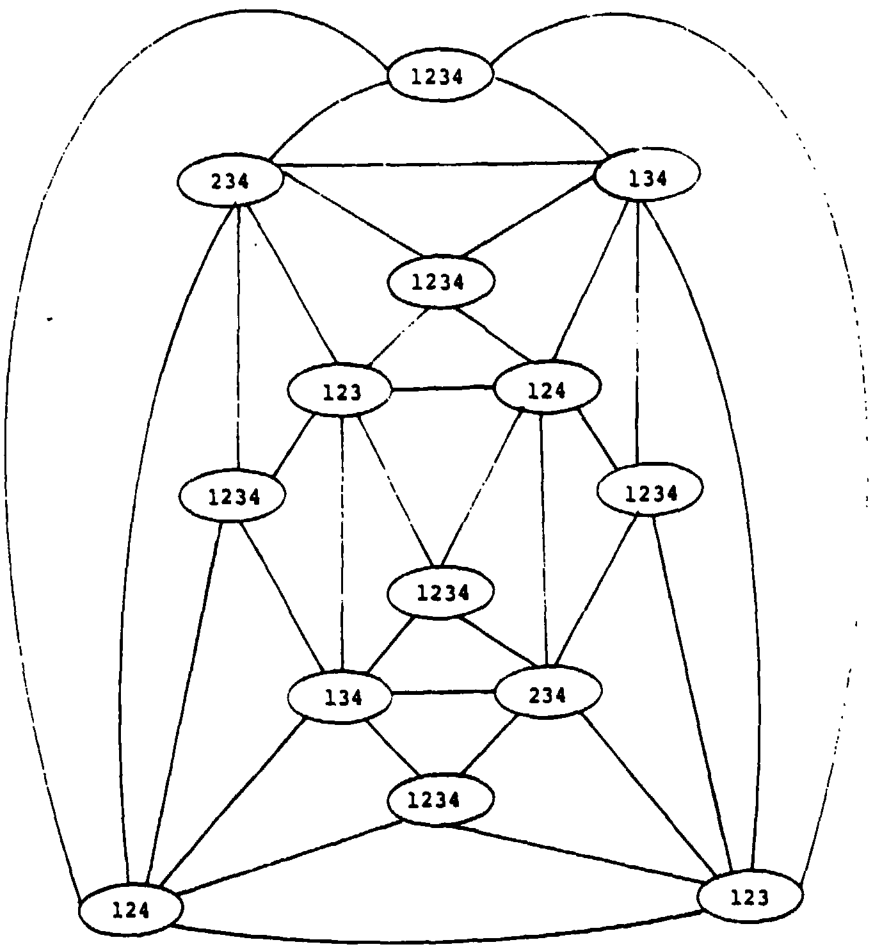


Figure 3-21. Not 3-choosable

3.11 (a:b) - Choosability

Suppose a is the number of distinct letters on each node, put there by the adversary, and we want to choose a b-subset from each node, keeping the chosen subsets disjoint whenever the nodes are adjacent. G will be (a:b)-choosable if such a choice can be made no matter what letters the adversary puts.

In terms of (a:b)-choosability we can say that if there does exist a planar bipartite graph which is not 3-choosable, it will have been a very close call in the following sense. If $a/b < 3$, then there exists a planar bipartite graph which is not (a:b)-choosable. In fact $K_{2,(\frac{a}{b})^2}$ will be not (a:b)-choosable.

Open Question

If G is (a:b)-choosable, does it follow that G is (am:bm)-choosable?

Open Question

If G is (a:b)-choosable, and $\frac{c}{d} > \frac{a}{b}$, does it follow that G is (c:d)-choosable?

Composition Lemma

Suppose H is obtained from G by adding edges. Let S be the subgraph consisting of those edges and their nodes.

If S is (d:a)-choosable, and G is (a:b)-choosable, then H is (d:b)-choosable.

<u>Proof</u>

Let the adversary put d letters on each node. First make a choice consisting of an a-subset from each node, with disjoint a-subsets on S-adjacent nodes. This first choice can be made because S is (d:a)-choosable. Next make a choice consisting of a b-subset from each node's a-subset, with disjoint b-subsets on G-adjacent nodes. This second choice can be made because G is (a:b)-choosable. The resulting choice makes the b-subsets disjoint on adjacent nodes of H. Thus H is (d:b)-choosable.

<u>Corollary</u>

If H is <u>not</u> 2k-choosable, and G is obtained from H by erasing disjoint edges, then G is <u>not</u> k-choosable.

Here is just one more theorem - a direct consequence of the fact that for given k and g there exists a family F of k-sets with the following three properties.

1. F does <u>not</u> have property B
2. For any two distinct $X,Y \in F$, $|X \cap Y| \leq 1$.
3. The smallest cycle has length g, in the graph which has nodeset = F, with an edge between nodes X and Y iff $|X \cap Y| = 1$.

For a proof of the above fact, please refer to P. Erdos and A. Hajnal, "On Chromatic Numbers of Graphs and Set Systems", Acta Math. Acad. Sci. Hungar., 17 (1966), pp. 61-99.

Theorem

For given k and g, there exists a bipartite graph G such that the smallest cycle in G has length >g, and choice #G > k.

Proof

Let F be a family of 2k-sets with properties 1., 2., 3. above. Let H be the bipartite graph having the 2k-sets of F as top nodes, and likewise as bottom nodes, with an edge between a top node X and a bottom node Y iff $X \cap Y \neq \Phi$.

First observe that H is not 2k-choosable, because F does not have property B. Any choice including one from each top node would use all the letters belonging to some 2k-set on the bottom.

Next obtain G by erasing those edges of H which connect two nodes having the same 2k-set. Thus G will inherit from F the property of having smallest cycle length >g.

The corollary to the composition lemma tells us that G is not k-choosable.

CHAPTER 4

THE M-PIRE PROBLEM

The m-pire problem was started by Heawood in the same 1890 paper [12] in which he exposed a flaw in Kempe's "proof" of the 4-color conjecture. Heawood proved, then, that the m-pire chromatic number of the sphere will be at most 6m, for every positive integer m. It means that for any map c empires on the sphere, in which each empire has at most m parts, 6m colors will be sufficient to give the same color to all parts of each empire, while requiring that any two empires get different colors if any part of one touches any part of the other along a border. In these words, the 4-color conjecture would say that the 1-pire chromatic number of the sphere is equal to 4.

To prove the necessity of 12 colors in the 2-pire case, Heawood drew the example shown here as Figure 4-1 Each of the empires named 1,2,...,12 has two parts, and each of the twelve touches all the others. It was, he said, "obtained with much difficulty in a more or less empirical manner". He remarked on his inability to find any regular or symmetric arrangement, and also said, "what essential variety there might be in such an arrangement of 12 two-division countries, as it exemplifies, is a curious problem, to which the one figure obtained does not affort much clue".

In 1974 I found a different 2-pire configuration -- before I knew about Heawood's -- but mine was just as irregular as his. Later, while struggling with the 4-pire case, I discovered a reasonably symmetric version -- it can be obtained by shrinking lettered faces to points in the "left hemisphere" of Figure 4-4. Meanwhile, quite independently, Scott Kim found the most symmetric 2-pire configuration of all, shown in Figure 4-2.

That the m-pire chromatic number of the sphere would be equal to 6m, for m > 1, was the implicit conjecture in Heawood's discussion. In fact, for m > 2, he clearly thought that there would always exist a configuration of 6m m-pires each touching all the others, thereby proving the necessity of 6m colors. As far as I know, this problem has remained open until now for m > 2. Figure 4-3 shows the necessity of 18 colors for the 3-pire case. Figure 4-4 shows 24 4-pires named 1,2,...,12 and A,B,...,L, each touching all the others, and thereby requiring 24 colors. Apparently the existence of an m-pire configuration requiring 6m colors is still an open problem for m > 4.

A digression is needed to discuss a problem posed by Gerhard Ringel in his 1959 book [13]. Suppose we draw a map of 2-pires on two separate spheres, requiring that each

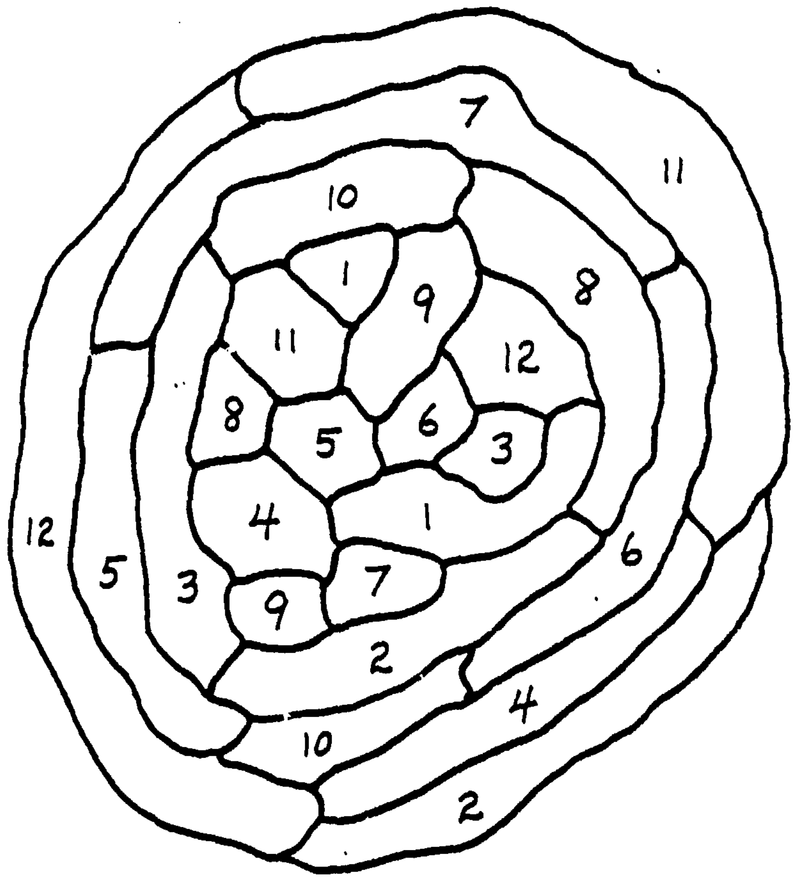


Figure 4-1. Heawood's Configuration

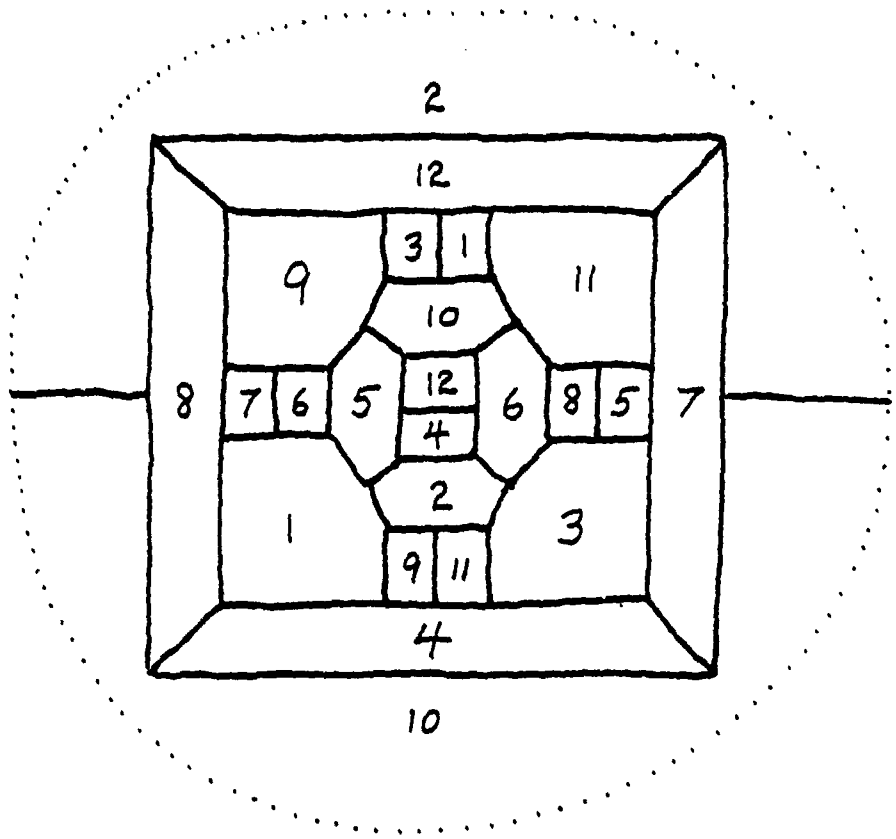


Figure 4-2. Scott Kim's Configuration

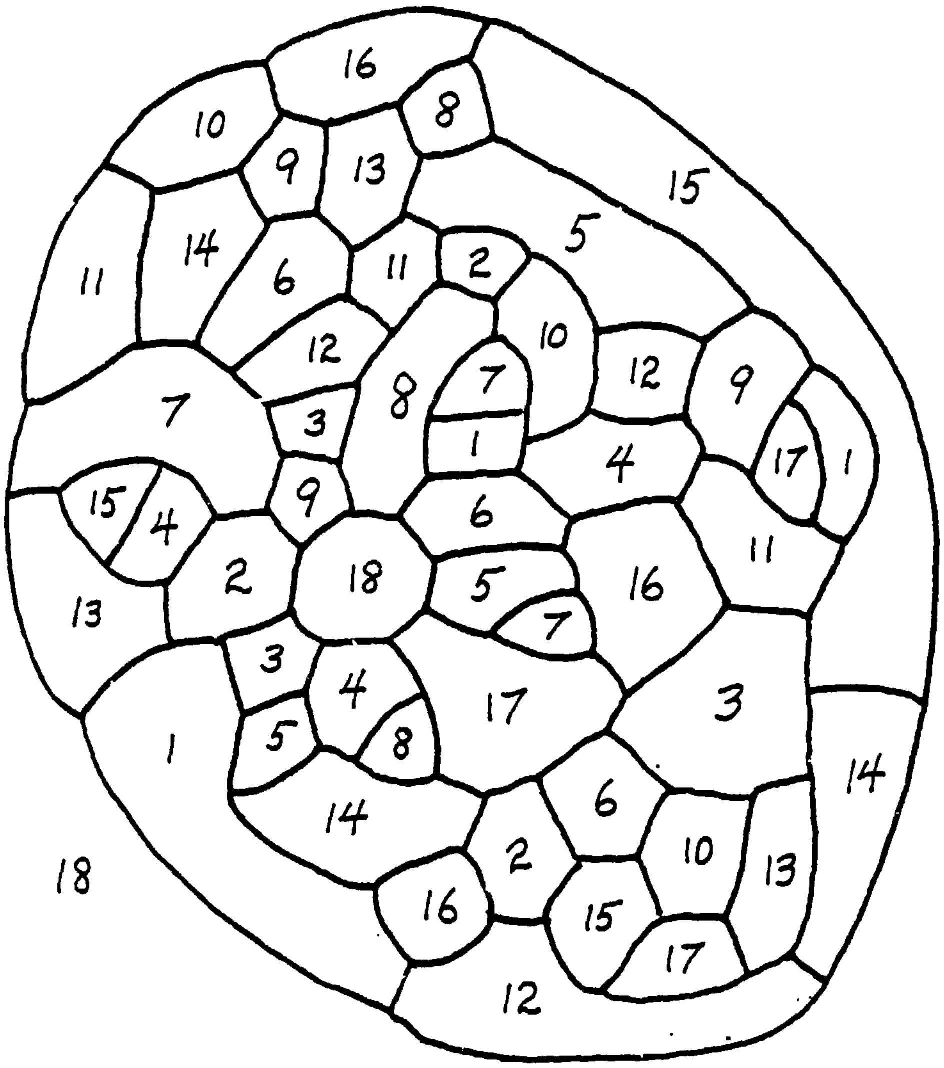


Figure 4-3. 3-pire on the Sphere

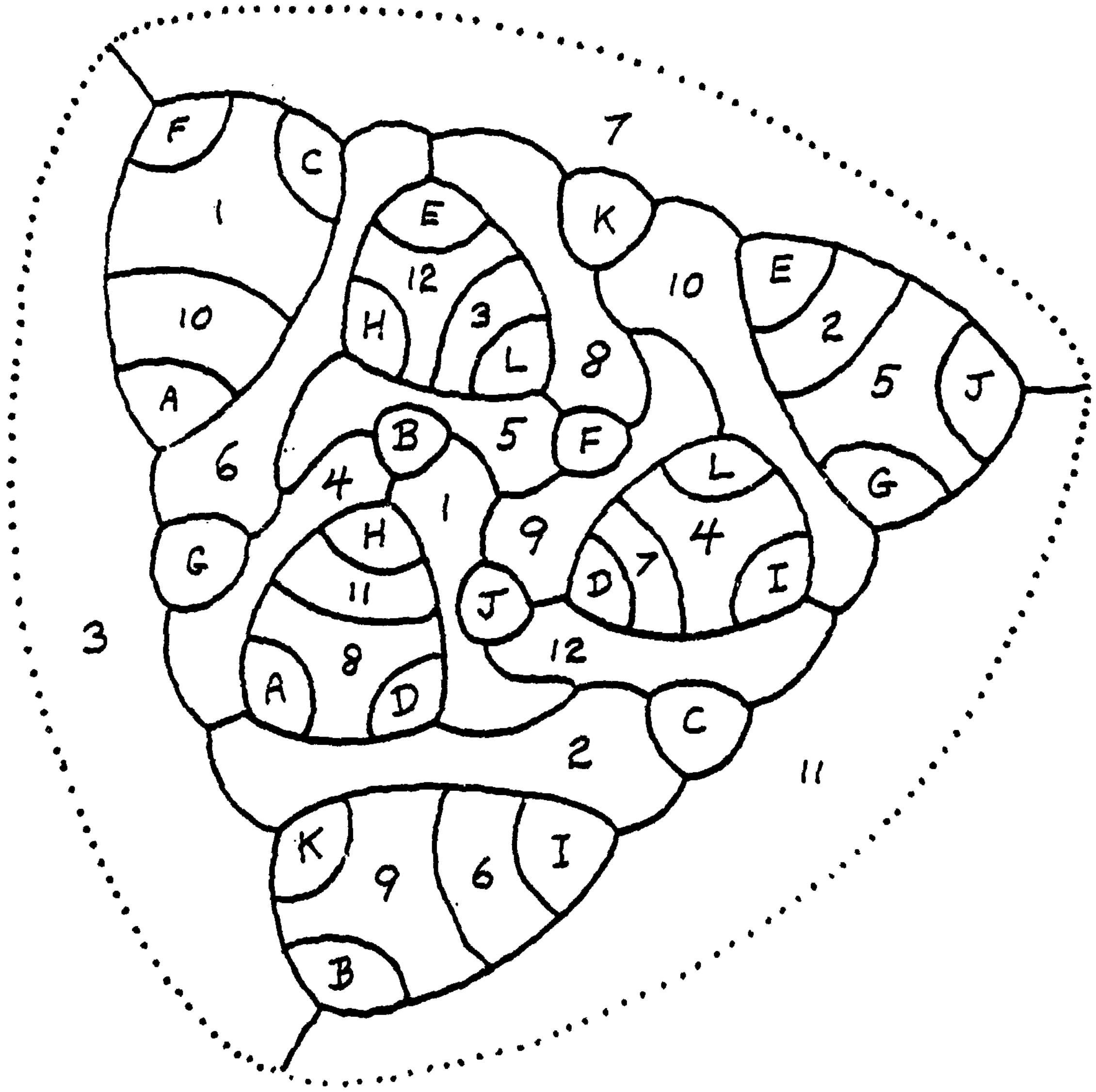


Figure 4-4a. 4-pire on the Sphere - Left Hemisphere

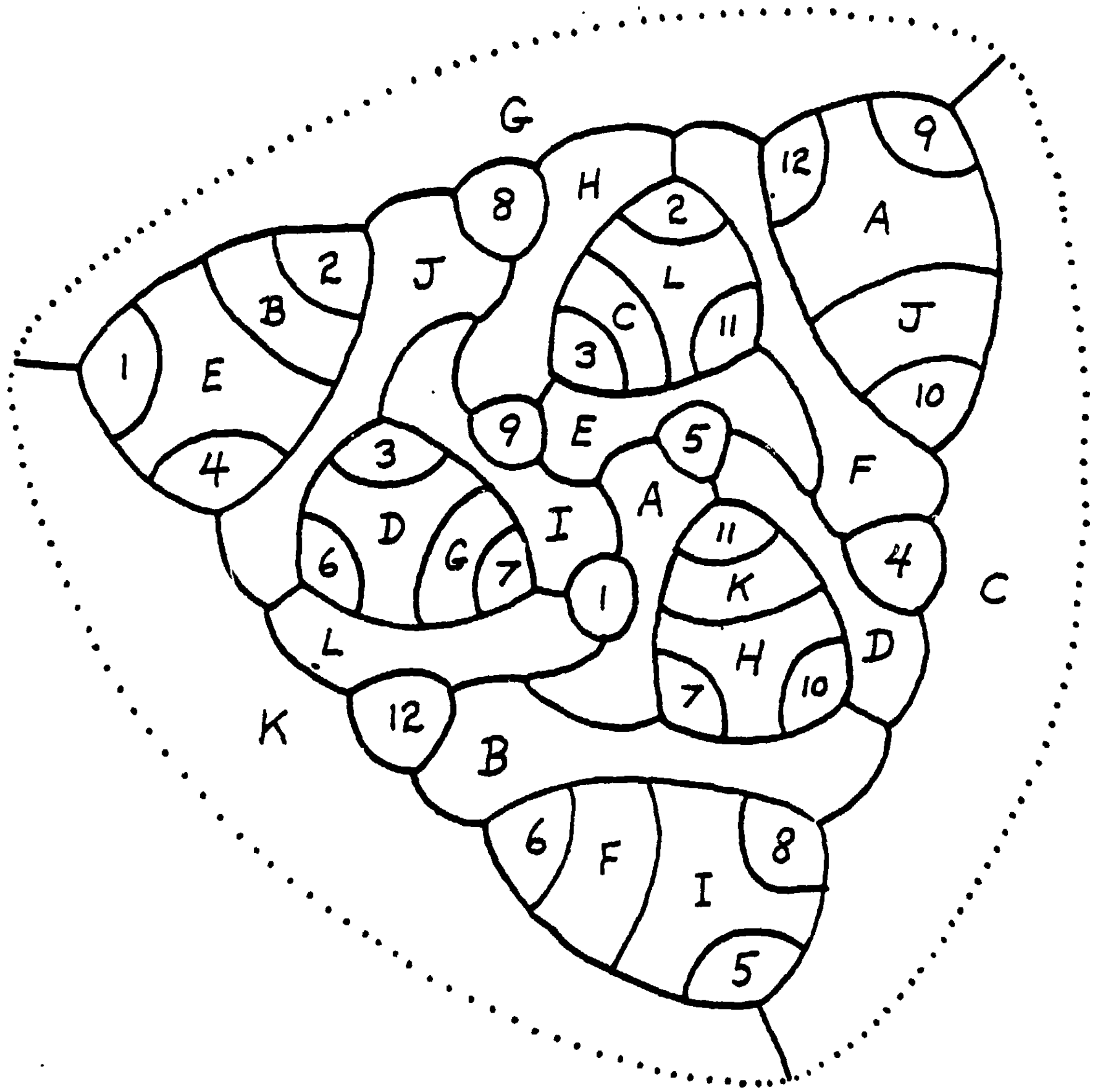
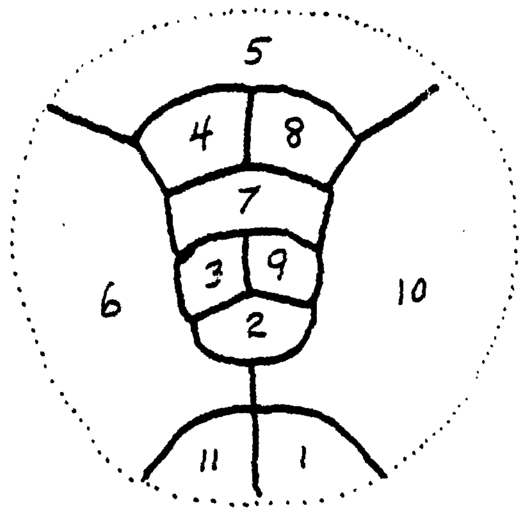


Figure 4-4b. 4-pire on the Sphere - Right Hemisphere



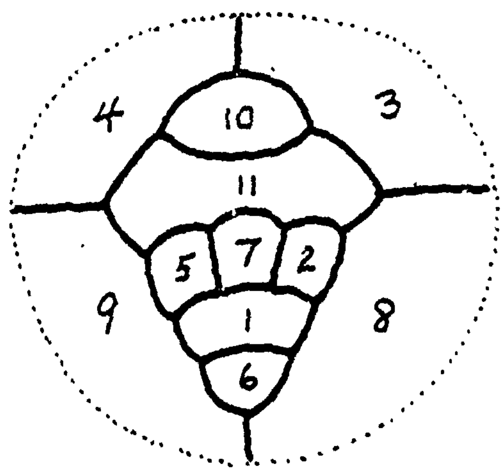


Figure 4-4c. Thom Sulanke's Configuration

empire have one of its two parts on one sphere, and one on the other. How many colors can such a map require? Well, the answer is only known to be 9 or 10 or 11 or 12.

Such a map on two spheres could not include nine 2-pires all touching each other, as was proved by Battle, Harary, and Kodama in 1962. In 1974 Thom Sulanke sent a remarkable picture to Ringel. Sulanke's configuration shows a map on two spheres requiring nine colors -- stupendous news to anyone familiar with the problem.

Returning to Figures 4-4 and 4-5, they more than show that the 4-pire chromatic number of the sphere is 24. They show a map of 4-pires on two spheres, each having two of its four parts on one sphere, and two on the other. Such a configuration, for even m, would exactly fit the limiting conditions which can be derived from Euler's surface formula. This writer offers the conjecture that, for positive even m, there will exist a map of 6m m-pires on m/2 spheres, each having two of its m parts on each sphere, with each m-pire touching all the others.

The m-pire chromatic number of the torus was shown by Heawood [12] to be at most 6m+1. In 1965 Ringel [18] gave a construction putting the complete graph on 6m+1 nodes on m separate torii in m-pire fashion. The remarkable thing about it was that his construction left the question open for the m-pire chromatic number of a single torus.

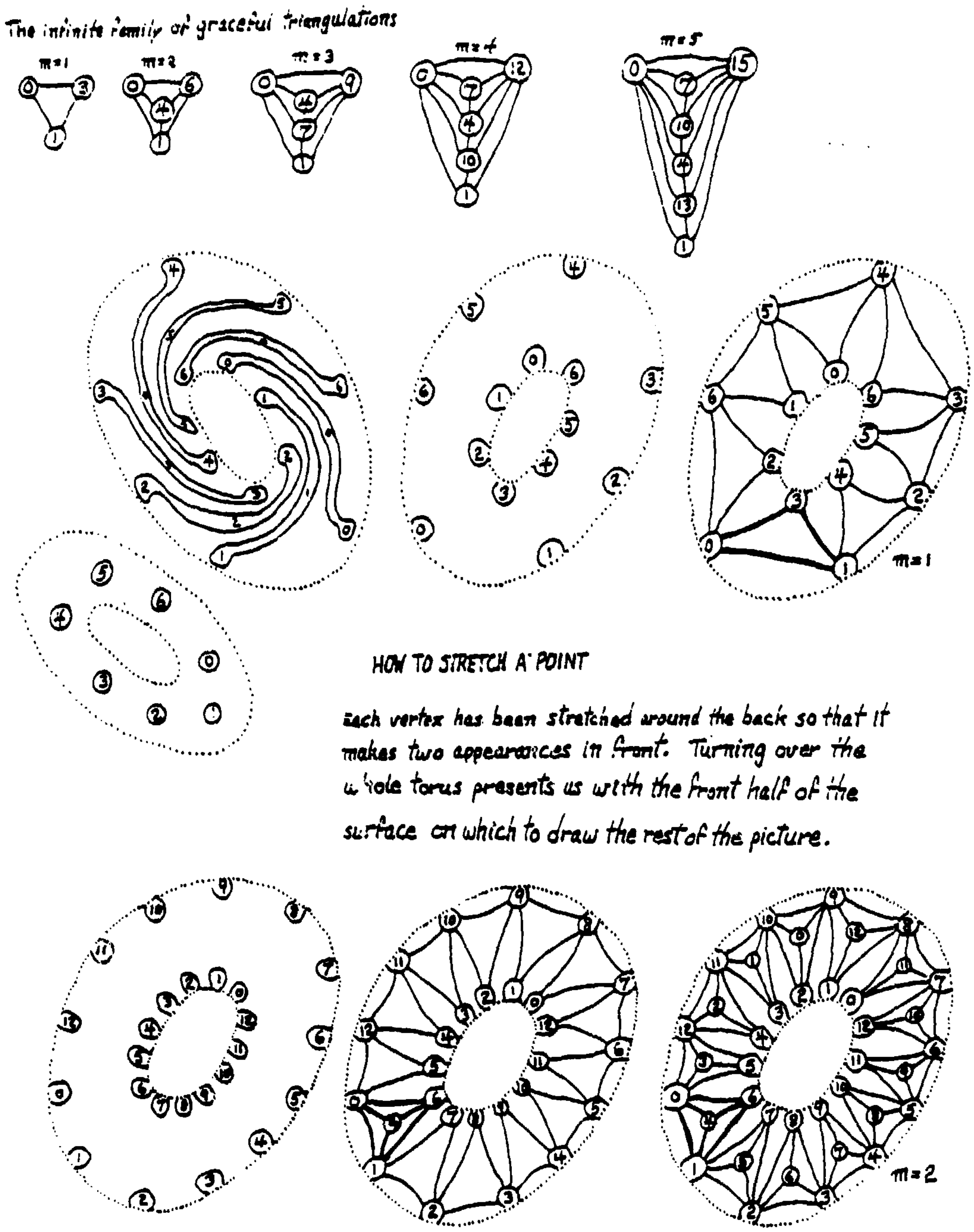


Figure 4-5. m-pire on the Torus - m=1 and m=2

In 1974 this writer found a construction method showing that the m-pire chromatic number of the torus is equal to 6m+1 for all m. It depends on the existence of at least one triangulation of the sphere which is graceful mode 6m+1, and has m+2 nodes. Just such a family was included in the article "How To Number a Graph" by Solomon W. Golomb, thus providing the key to the torus problem All the graphs in that family are graceful, and a fortiori graceful mod 6m+1, as shown in Figure 4-5.

Figures 4-6 and 4-7 illustrate the construction for m = 3 and m = 4. The complete graph with 6m+1 nodes is put on the torus in m-pire fashion with each node represented by m vertices on the surface. To reform the torus the dotted borders just need to be rejoined so that the two halves of each border vertex agree.

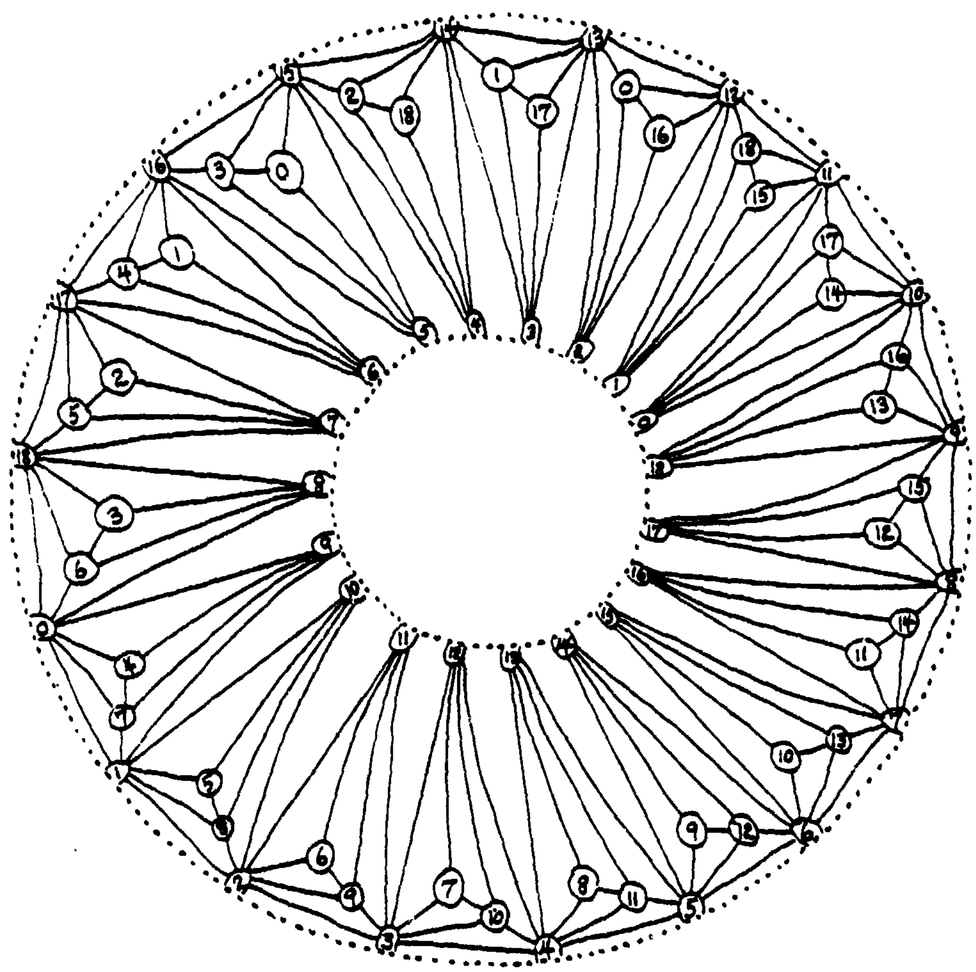


Figure 4-6. m-pire on the Torus - m=3

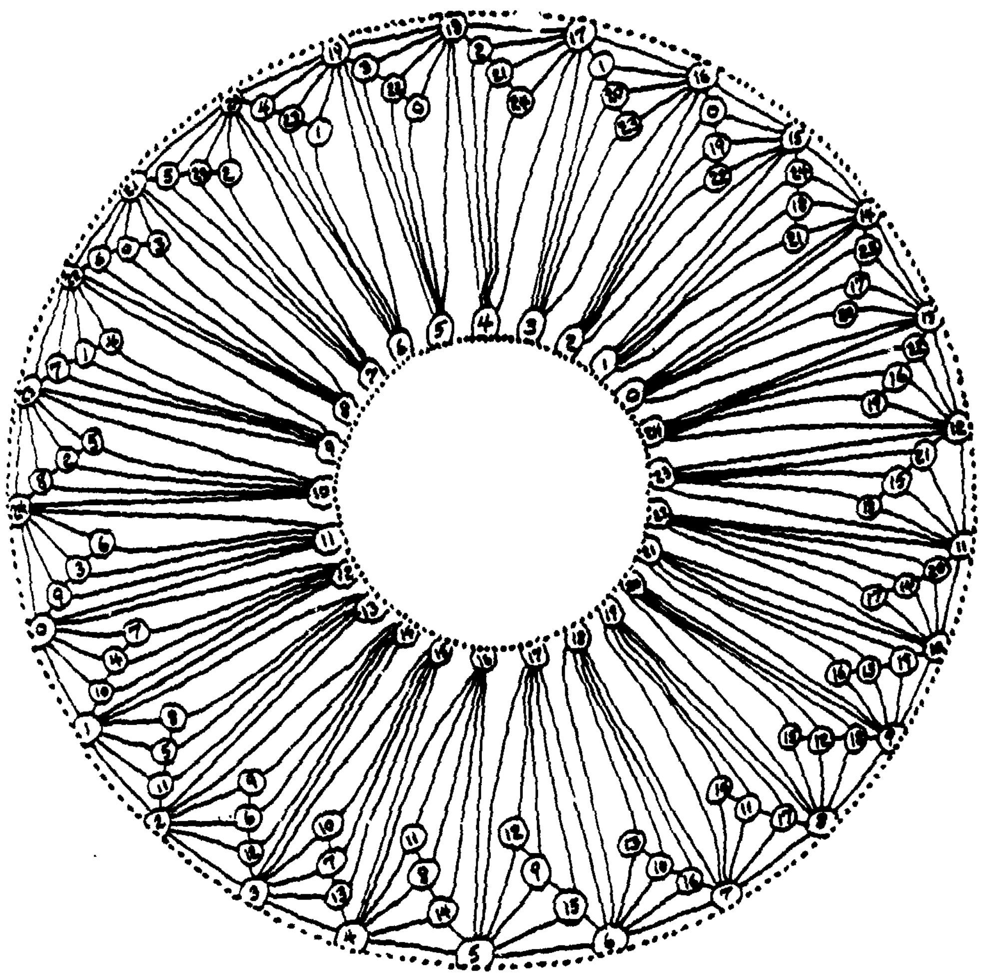


Figure 4-7. m-pire on the Torus - m=4

CHAPTER 5

TWO DIMENSIONAL SYNCHRONIZATION PATTERNS FOR MINIMUM AMBIGUITY

5.1 Summary

There are numerous problems arising from radar, sonar, physical alignment, and time-position synchronization, which can be formulated in terms of finding two-dimensional patterns of ones (dots) and zeroes (blanks) for which the two-dimensional (spatial) autocorrelation function, the so-called "ambiguity function" of radar analysis, has minimum out-of-phase values (or "minimum sidelobes"). A typical context is one in which it is desired to produce a sequence of distinct frequencies ("tones") in consecutive time slots, so that if a returning echo of this sequence is shifted in both time and frequency by a moving target, the only translate of the original pattern having high correlation with the received configuration will be the one whose time shift corresponds to the correct range, and whose frequency shift to the correct velocity, of the target.

In this chapter, a number of closely related combinatorial problems, corresponding to specific assumptions about the type of time-frequency sequence which may be

appropriate in a particular application, are formulated in terms of square or rectangular arrays of dots, with appropriate constraints on the two-dimensional correlation function. The current state of knowledge concerning each of these problems is summarized. It is hoped that more general constructions may be found, leading to larger families of solutions, as well as better computational algorithms for finding individual solutions which may lie outside of the general families.

These problems may be regarded as the two-dimensional analog of the one-dimensional "ruler problems", described at length in [19], which have application to onedimensional synchronization and alignment problems, and to radar or sonar situations in which the doppler shift can be neglected. An example of a two-dimensional problem corresponding to a square array configuration which arose in the context of a practical sonar problem has been described by John C. Costas in [20] and [21].

5.2 Introduction

We consider patterns of dots in a rectangular grid under different combinations of requirements. The unifying concept is that of a pattern which will give major agreement with shifted copies of itself only when these are in special positions, and otherwise only minor agreement. In fact, our basic patterns have the property that in any position reachable by horizontal and vertical shifting, other than the original position, the pattern will overlap with the original in at most one dot location.

FigureS-1 shows an example in which the number of dots is maximized for a 3×3 array.

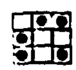


Figure 5-1. An Optimum 3×3 Array

Another example in which the number of dots is maximized, for a 5×5 array, is shown in Figure 5-2.

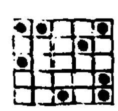


Figure 5-2. An Optimum 5×5 Array

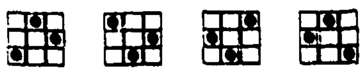
These "minimum agreement" patterns may be viewed as a generalization to 2 dimensions of the "ruler problems" which have numerous applications to radar, sonar, synchronization, crystallograhy, etc. [19]. For example,

 minimizes n in a $1 \times n$ array with 4 dots.

5.3 Constellations

Two-dimensional agreement patterns of a special kind, arising from a practical sonar problem, were suggested by John Costas, who asked about an $n \times n$ pattern of n dots with one dot in every row and column having the property that any horizontal and vertical shifting would result in at most one dot position agreement. The original application of these "constellations" was to a sonar problem [20], but applications to radar and to synchronization and alignment also exist. Constructions by L. Welch and A. Lempel show that such constellations exist for infinitely many values of n. In fact, there are effective algebraic constructions for $n = p-1$ and $n = p^m-2$ whenever p is prime and m is any positive integer, and in certain other cases as well.

On the other hand, we do not have any construction method, effective or otherwise, for general n. In relation to the problem of a general construction, we conjecture that the probability of finding a constellation by random search will go to zero as n goes to infinity. For example,



3x3 constellations

3x3 non constellations.

Figure 5-3. The Constellations and Non-constellations when n=3

as shown in Figure 5-3, regarding an n×n array as a permutation of n objects (namely, the column indices corresponding to the successive row indices), the probability of choosing a constellation at random for $n = 3$ is $\frac{4}{6} = \frac{2}{3}$. However, the probability drops to $\frac{192}{5040} = \frac{4}{105}$ for $n = 7$ (determined by an exhaustive search of this case).

Another open combinatorial question, posed by S.W. Golomb, is whether or not any "queen constellations" exist. As already described, constellations are necessary configurations of non-attacking rooks. Do any exist which are also configurations of non-attacking queens?

For both the probability question and the problem of trying to enumerate all constellations, there is the obvious objective of trying to find a good computer algorithm to extract just the constellations from among all the permutations of n symbols.

known results and constructions for constellations are presented in greater detail in Sections 5.8 and 5.9.

5.4 Sonar Sequences

As observed by H. Greenberger, the application to doppler sonar or radar does not require the restriction to one dot per row - only the restriction to one dot per column. The pattern can be read like music notation giving a sequence of tones, but with only one tone at each beat. When the "tones" return after being reflected from a moving target, horizontal shift will correspond to elapsed time and vertical shift will correspond to doppler. The number of rows will be limited by the context, but generally the number of columns will be what we wish to maximize. Figure 5-4 shows an example with 4 tones which can extend to 8 beats.

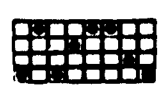


Figure 5-4. A 4×8 "Sonar Sequence"

It is easy to prove that with n rows, the maximum number of columns is at most 2n. However, the above example (Figure 5-4) of a sonar sequence which is 4×8 may turn out to be the last case where n 2n is actually achievable. At least, it is the largest such case discovered to date.

Our sonar sequence problem is to maximize m in an n×m pattern where n is given. The largest example obtained thus far is only 13×18. Figure 5-5 shows a 10×14 example, which incidentally also illustrates the "palindromic method" of construction (used because it is easier to check).

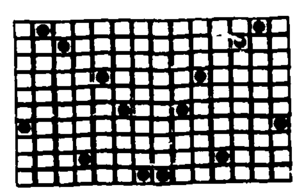


Figure 5-5. A 10×14 Sonar Sequence

As m may in fact get closer to n than to 2n as n increases, there is another possible approach, suggested by H. Greenberger, for constructing long sonar sequences. Specifically, we could change the requirement to allow agreement of two dots but never three dots. Then instead of a ratio of m:1 we could seek the best m:2 configuration. For example, Figure 5-6 shows a 3×10 pattern which can never be shifted from its original position to cause more than a two-position agreement of dots.

Figure 5-6. A 3×10 Sonar Sequence with a Sidelobe Bound of 2

We can still prove that under these liberalized conditions, $m_o:2 \leq 2n:1$, where "m_o" denotes the largest possible value of m for the given value of n. It seems quite probable, however, that for larger n these revised sonar sequences may achieve a better ratio than with a sidelobe value limited to a single agreement of dots. (A corresponding likelihood exists for the "radar sequences": discussed below.)

5.5 Radar Sequences

If the practical application does not require doppler measurement, then the min-agreement patterns may achieve still higher ratios. A radar imaging device might use either a tonal sequence, or a pattern with one dot per column and a maximum number of columns designed to agree in at most one dot position after all horizontal shifts, not caring about vertical shifts.

In this case, with n rows, when we have an nx max "radar sequence", we can prove that $2n \leq \max \leq 3n$.

For example, in Figure 5-7 we see a pattern for which $n = 3$ and $max = 7$.

Figure 5-7. A 3×7 "Radar Sequence"

5.6 Some General Problems

a. We can establish a connection between 1-dimensional ruler problems, which have been extensively studied (cf. [19], and some of these 2-dimensional min-agreement patterns, by the method of shearing, as illustrated in Figure 5-8.

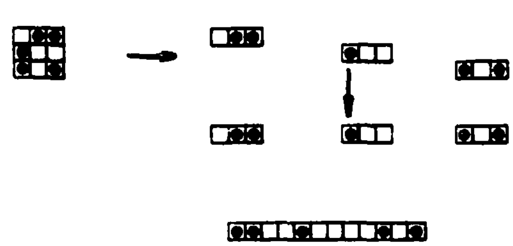


Figure 5-8. The Method of Shearing, Used to Obtain a 1-Dimensional Ruler from a 2-Dimensional Array

(Efficient mappings in the reverse direction would be quite useful)

b. Another problem is to find a pattern with the <u>minimum</u> number of dots in an a×b rectangule such that no additional dots can be placed without causing a repeat pair.

c. Going to 3-dimensions (or 4-dimensions), we can ask for the maximum number of dots in an a×b×c region (a×b×c×d region) having no repeat vector differences between pairs of dots. Figure 5-9 shows several simple 4-dimensional examples.

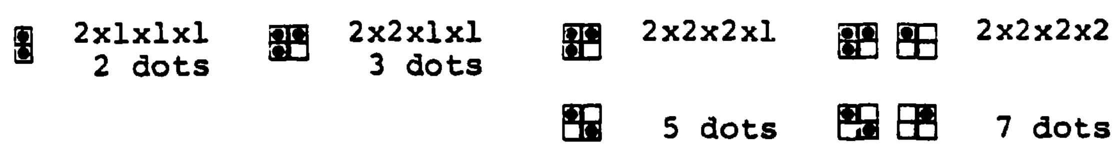


Figure 5-9. Some Simple Examples of the 4-Dimensional Generalization

5.7 <u>Sum Distinct Sets</u>

The unifying idea in most of these problems is to consider some limited region in a vector space or "module" over the integers, and ask for the maximum number of vectors which can be positioned in the region in such a way that all pairwise sums are distinct. Two algorithms for verifying the sum-distinct property are illustrated on the following page:

a. Difference Triangle Algorithm

2 3 1 1 3 2

1 -2 0 2 -1

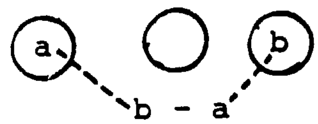
-1 -2 2 1

1 0 1

1 -1

0

[make sure rows have no repeated signed differences]



b. Sum Triangle Algorithm

(1,2) (1,3) (2,1) (3,1) (3,3)

[2,4] (2,5) [2,6] (3,4) [4,2] (5,2) [6,2] (6,4) [6,6]

(3,3) (4,4) (5,4)

(4,3) (4,6)

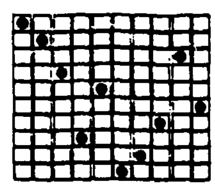
(4,5)

[all vector sums distinct]

5.8 Known Constructions for n×n Costas "Constellations"

a. Construction 1. (L.R. Welch). Let p be a prime number, with $n = p-1$. Pick a "primitive root" g modulo p. Put a dot in the cell (i,j) of the n×n array if and only if $j \equiv g^i \pmod{p}$, where $1 \le i \le n$ and $0 \le j \le n-1$.

Example: $p = 11$, $g = 2$. The powers of g modulo 11 are 1 2,4,8,5,10,9,7,3,6. Thus the array is as shown in Figure 5-10.



p = 11

Figure 5-10. A 10×10 Constellation Using the Welch Construction

b. Modification of Construction 1:

A. Omitting the top row and the left-most column, an n×n constellation is always obtained with $n = p-2$. (See the example shown with $p = 11$.)

B. In the case that $g = 2$ (only certain primes, including 3,5,11,13,19,29... in a subsequence of the primes which is believed to be infinite, have 2 as a primitive root), another row from the top and another column from the left may be removed to obtain an n×n constellation with $n = p-3$. (See the example again with $p = 11$.)

c. <u>Construction 2</u>: (A. Lempel). Let q be any power of any prime number. Let α be a primitive element of GF(q), the field of q elements. We obtain a <u>symmetric</u> $n \times n$ constellation with $n = 1-2$, by the rule that we put a dot in the cell (i,j) if and only if $\alpha^i + \alpha^j = 1$ in GF(q). Here $1 \leq i \leq n$ and $1 \leq j \leq n$.

<u>Example</u>: Let $q = 8$, and let α satisfy $\alpha + \alpha^3 = 1$. Then also $\alpha^2 + \alpha^6 = 1$ and $\alpha^4 + \alpha^5 = 1$. Then in the 6×6 array shown in Figure 5-11, dots appear at (1,3), (2,6), (4,5), and their reflections (3,1), (6,2), (5,4).

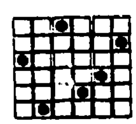


Figure 5-11. A 6×6 Constellation, Using the Lempel Construction

(Note that while an $n = 6$ example also occurs from Construction 1, it will not be a symmetric example.)

d. <u>Modification of Construction 2</u>:

A. If q is an odd prime, and 2 is primitive mod q, then taking $\alpha = 2$, Construction 2 yields an example with a dot in the lower right-hand corner. Removing the bottom row and the right-most column yields an n n example with n = q-3. (While this is the same size as the result of Modification B to Construction 1, it is in general a different, and symmetric, example.)

e. Summary of Values with Known Constructions

In Table 5-1, a summary is presented, for $n \leq 100$, of the known constructions which lead to an n×n constellation. The constructions are identified in the Table as 1, 1A, 1B, 2, and 2A, respectively, and the values of p or q being used are identified. If there is a blank next to a given value of n, this need not mean that no examples exist, but merely that the specific constructions described herein do not lead to examples.

5.9 Complete Enumeration of Small Constellations

An exhaustive enumeration of the Costas constellations which are inequivalent under the dihedral symmetry group of the n×n square has been carried out through n = 7. The results for $n \leq 6$ are shown in Figure 5-12.

Queen constellations do not exist for $n \leq 10$, as we have discerned by surveying all configurations of n non-attacking queens on an n×n board for $n \leq 10$. These

Table 1

Summary of Values n for which These Constructions Yield an n×n Example

n	Constructions	n	Constructions
1	TRIVIAL	41	1A(p=43), 2(q=43)
2	1(p=3), 1B(p=5), 2(q=4)	42	1(p=43)
3	1A(p=5), 2(q=5)	43	
4	1(p=5)	44	
5	1A(p=7), 2(q=7)	45	1A(p=47), 2(q=47)
6	1(p=7), 2(q=7)	46	1(p=47)
7	2(q=9)	47	2(q=49)
8	1B(p=11), 2A(q=11)	48	
9	1A(p=11), 1(q=11)	49	
10	1(p=11), 1B(p=13), 2A(q-13)	50	1B(p=53), 2A(p=53)
11	1A(p=13), 2(q=13)	51	1A(p=53), 2(q=53)
12	1(p=13)	52	1(p=53)
13		53	
14	2(q=16)	54	
15	1A(p=17), 2(q=17)	55	
16	1(p=17), 1B(p=19), 2A(q=19)	56	1B(p=59), 2A(q=59)
17	1A(p=19), 2(q=19)	57	1A(p=59), 2(q=59)
18	1(p=19)	58	1(p=59), 1B(p=61), 2A(q=61)
19		59	1A(p=61), 2(q=61)
20		60	1(p=61)
21	1A(p=23), 2(q=23)	61	
22	1(p=23)	62	2(q=64)
23	2(q=25)	63	
24		64	1B(p=67), 2A(q=67)
25	2(q=27)	65	1A(p=67), 2(q=67)
26	1B(p=29), 2A(q=29)	66	1(p=67)
27	1A(p=29), 2(q=29)	67	
28	1(p=29)	68	
29	1A(p=31), 2(q=31)	69	1A(p=71), 2(q=71)
30	1(p=31), 2(q=32)	70	1(p=71)
31		71	1A(p=73), 2(q=73)
32		72	1(p=73)
33		73	
34	1B(p=37), 2A(q=37)	74	
35	1A(p=37), 2(q=37)	75	
36	1(p=37)	76	
37		77	1A(p=79), 2(q=79)
38		78	1(p=79)
39	1A(p=41), 2(q=41)	79	2(q=81)
40	1(p=41)	80	

n	Constructions	n	Constructions
81	1A(p=83)	91	
82	1(p=83)	92	
83		93	
84		94	
85		95	1A(p=97), 2(q=97)
86		96	1(p=97)
87	1A(p=89), 2(q=89)	97	
88	1(p=89)	98	1B(p=101), 2A(q=101)
89		99	1A(p=101), 2(q=101)
90		100	1(p=101)

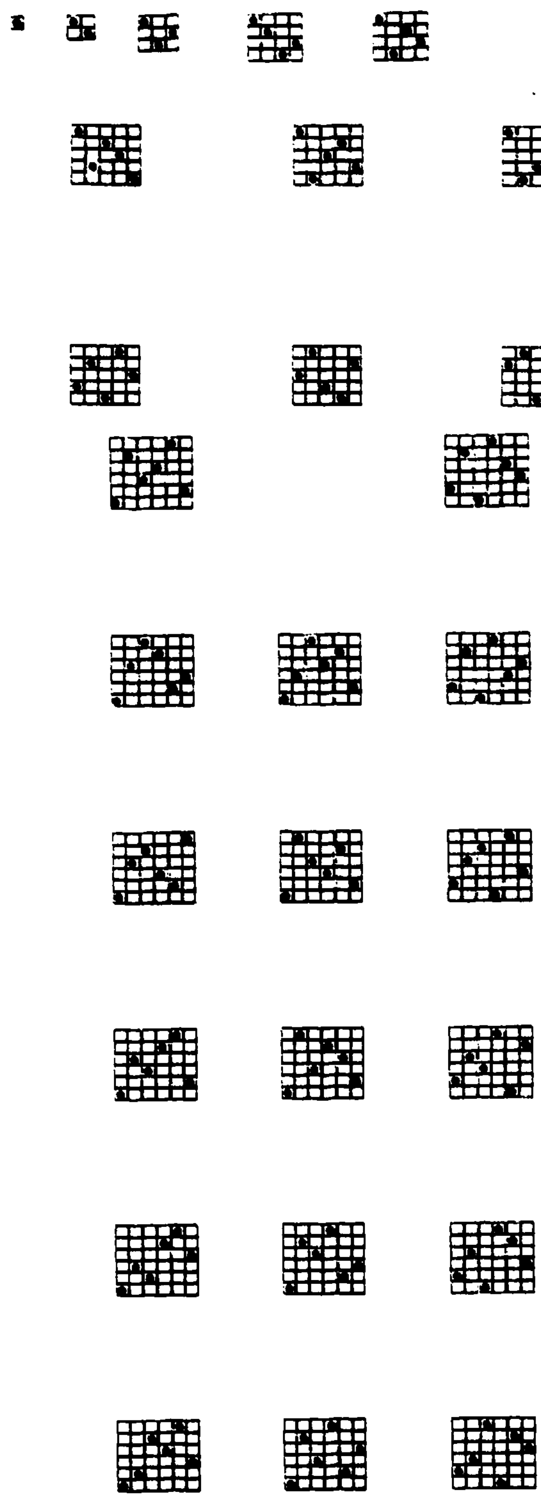


Figure 5-12. The "Costas Arrays" Inequivalent Under the Dihedral Symmetry Group D_4 of the Square, for $n \leq 6$

were published in 1891 by Eduard Lucas in his book "Récréations Mathematiques".

APPENDIX

THE TERMINOLOGY OF GRAPH THEORY

The primary concern of graph theory, as I understand it, is the art of picturing a problem. However, "Graph Theory" has acquired a restricted meaning in recent decades so that much work has been concentrated on "pictures" which fit a narrow description. To a large extent the pictorial descriptive devices have themselves become the subject matter of abstract study.

The now voluminous literature of graph theory often uses two or three terms for the same thing, and contains a confusing multitude of defined notions. It is with the hope that these five chapters will appear as elementary as they really are, that just the terminology used here is covered in the following informal discussion.

GRAPH	A (simple) graph consists of a set of
NODES	objects called nodes, together with a
	set of 2-subsets of the nodes called
EDGES	edges. A graph is completely specified
	when it is told what the nodes are,
	and which pairs of nodes are edges.
	To make a picture of a given graph we

can think of each node as a spot, or vertex, or point, and think of each edge as wire, or arc, or line segment connecting its pair of nodes. Two

ADJACENT distinct nodes are adjacent, or joined, if there is an edge to which they both belong.

PATH A path is a sequence $e_1, e_2, \ldots, e_k$ of distinct edges such that $e_1 \cap e_{i+1} \neq 0$ for each i from 1 to k-1. If a path begins and ends on the same node

CIRCUIT it is called a circuit. Paths and circuits are thus allowed to repeat nodes, but not edges. A circuit (with $k > 2$) which furthermore does not

CYCLE repeat nodes is called a cycle.

CONNECTED A graph is connected if it contains a path between any two distinct nodes. A graph which is connected but contains

TREE no cycles is called a tree. If a graph contains no cycles, then its connected

COMPONENTS components are trees, so naturally it

FOREST is called a forest.

	If G and H are graphs, all nodes of G are nodes of H, and all edges of G
SUBGRAPH	are edges of H, then G is a subgraph of H. If furthermore all the edges of H on pairs of nodes of G are also
INDUCED SUBGRAPH	edges of G, then G is an induced subgraph of H. Thus each subset of nodes of H may have several subgraphs, but each subset of nodes determines exactly one induced subgraph.
VALENCE	The valence of a node is the number
DEGREE	of edges on it. "Valence" and "degree" mean the same thing. A graph is called
REGULAR	regular if all its nodes have the same valence.
COMPLEMENT $\bar{G}$	The complement $\bar{G}$ of a graph G is a graph with the same nodes as G, but each pair of nodes is an edge of $\bar{G}$ iff it is not an edge of G.
	The graph with n nodes which has $\binom{n}{2}$ edges (i.e., every possible edge is
THE COMPLETE GRAPH K_n	present) is usually denoted by K_n. It is called the complete graph.

COMPLETE BIPARTITE GRAPH $K_{a,b}$

If the nodes are partitioned into two sets A and B, with $|A| = a$ and $|B| = b$, and the graph has an edge on two nodes iff one is in A and the other is in B, then the graph is the complete bipartite graph denoted by $K_{a,b}$.

Here are two examples of applications.

1. In the Leech tree of chapter 1 the five edge numbers could be the values of resistors. Then the tree would provide an efficient resistance standard, giving resistances 1,2,...,15.

2. As pointed out by Robert Scholtz, a choosability problem could arise in a communication network when some pairs of terminals are forbidden to use the same frequency. A degree of freedom could be gained by only having to require that each terminal have a specified number of frequencies available (not which frequencies - only how many), if we knew that a nonconflicting choice could be made at any time.

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